CAPITAL BUDGETING OBJECTIVE FUNCTIONS THAT CONSIDER DIVIDENDS AND TERMINAL WEALTH

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NOMENCLATURE

a_j — present worth computed at the horizon year T of all the cash flows of the project j occurring after year T.

a_{tj} — cash flow associated with project j occurring within the horizon period.

A — matrix of coefficients of the variables within the constraints of the nonlinear model.

b — column vector of the right hand side elements of the constraints of the nonlinear model.

b_1 — slope of the time series of dividend payments within the horizon.

c_1, c_2, c_3 — nonnegative parameters which assign weight to the utility terms in the objective functions of the proposed models.

$\bar{D}$ — average dividends paid within the horizon.

D_{min} — minimum annual dividend payment within the horizon.

D_t — dividend payment at year t.

f_t — dual variable of the linear model associated with the budget constraints (3.2) and (3.3).

g — dual variable of the linear model associated with constraint (3.4)

G — terminal wealth of the company at the end of the horizon.

r_{bt} borrowing rate of the money borrowed at t - 1 and paid at t.

$r_{\ell t}$ lending rate of the money lent at t - 1 and paid at t.

S set of accepted projects.

S' subset of S containing the fully accepted projects.

S'' subset of S containing the partially accepted projects.

s standard deviation of the stream of dividends paid within the horizon.

T horizon year.

$(u_M)_t$ dual variable of the nonlinear model associated with the budget constraints (3.50) and (3.51).

$(u_D)_t$ dual variable of the nonlinear model associated with the minimum dividend constraint (3.52).

h_t dual variable of the linear model associated with the minimum dividend constraint (3.5).

i dual variable of the linear model associated with the constraint (3.6).

k stockholders' discount rate.

ℓ_j dual variable of the linear model associated with the project constraint (3.8).

M_t money available from other sources for year t.

n number of projects.

q an r-component row vector containing the coefficients. cients of the linear terms of the objective function of the nonlinear model.

Q	an r x s matrix containing the coefficients of the nonlinear terms of the nonlinear objective function.
$(u_x)_j$	dual variable of the nonlinear model associated with project constraint (3.53).
v_t	money to be lent at year t.
w_t	money to be borrowed at year t.
x_j	decision variable for project j.

SUMMARY

Among the mathematical programming models in capital budgeting there are many which involve dividend payments within some certain planning period or horizon. These models try to determine a stream of dividends to satisfy the interests of management or the stockholders, or both. One of the more important characteristics of the stream is the steadiness of the dividends: ideally there should be only small fluctuations about an average dividend level. Also, current dividends should not be so excessive as to jeopardize the future financial position of the company. But many of these dividend models seem to be improperly formulated, and so they do not simulate correctly what we see in the real world.

The purpose of this thesis is to formulate and demonstrate capital budgeting models which deal with these issues in a more satisfactory manner than similar models formulated before. It is not intended to examine the problem of the valuation of the company. The objective of our models is to provide high overall dividends, balanced against steadiness of dividends and maintenance of terminal wealth.

Two models are formulated, one linear and one nonlinear, using slightly different approaches. The general approach used is to take Weingartner's classical horizon model and to add to the objective function some terms which represent the stockholders' and management's utilities for dividends and terminal wealth. This is accomplished by using linear as well as nonlinear programming. In each of the objective functions, three parameters give weights to the utility terms.

Through an extensive theoretical analysis, certain optimality conditions are determined and demonstrated with some numerical results for example problems. Using a complete 4^3 factorial experiment, a parametric analysis is made to investigate the behavior of both models over the parameter levels. The performance of the models is measured by the company's terminal wealth, average dividends paid within the horizon, and the standard deviation of the stream of dividends.

The linear model is found to have many drawbacks, but the nonlinear model achieves satisfactorily the objectives of this research.

CHAPTER I

INTRODUCTION

The development of highly sophisticated computer systems has allowed practicioners of financial planning and engineering economy to develop and use more and more detailed mathematical models in order to represent more realistically the different circumstances and conditions which the models try to simulate.

Also the concern of the large companies about risk and uncertainty has increased in the past 25 years stimulating more sophisticated capital budgeting techniques. Computer simulation, probability theory and PERT/criteria path techniques have been increasingly used in capital budgeting since the end of the fifties as Klammer reports (12). He also reports that linear programming is used as never before. However, an increase was not reported in the use of nonlinear programming, perhaps because his latest survey reported on the article of the reference was made in 1970, the date of the latest nonlinear programming reference included in Klammer's survey, nonlinear algorithms were not as efficient as now.

Among the mathematical programming models in capital budgeting, there are many which involve dividend payments within some certain planning period or horizon. These models try to determine the best stream of dividends which might be obtained by the stockholders. One of the more important characteristics of the stream is the steadiness of the dividends: small fluctuations about an average level.

For stockholders who prefer dividends rather than capital gains, this steadiness represents a nearly certain income, and they pay for obtaining it. However, the best tool so far for handling that steadiness within a capital budgeting model is an inclusion within the model of a non-decreasing dividend policy, which establishes that the dividend payment of a specific year should be greater than or equal to that of the year before. Although helpful, it does not simulate correctly what one observes in the real world.

The objective of obtaining a steady stream of dividends is not easy to obtain. The company might sacrifice earnings and reduce its terminal wealth in order to pay dividends and keep the same pay-out. It is an important thing to make the stockholders to feel satisfied with their dividends, but it is not clear how to balance that satisfaction against earnings from an additional investment.

Given the above, it is the purpose of this thesis to formulate and demonstrate capital budgeting models which deal with these issues in a more satisfactory manner than previously, It is not intended to examine the problem of the valuation of the company. The objective of any model will be to provide high overall dividends, balanced against steadiness of dividends and maintenance of terminal wealth.

The approach used in this thesis is to take Weingartner's classical horizon model (19), and to add to the objective function some terms which represent the stockholders' utility function for dividends and terminal wealth. This is accomplished by using linear as well as nonlinear programming.

In Chapter II, several outstanding mathematical models are analyzed. Some models do not take into account dividend payments, but given their characteristics, they are studied, as in the case of Weingartner's horizon model which is used as a basis for the present work. Another model which plays an important role here is that of Bernhard (4), whose concepts about a company's terminal wealth influenced this research.

In Chapter III, two models, one linear and one nonlinear are presented. There is included a complete analysis of the meaning of the additional objective function terms, as well as an optimality conditions analysis. Each model contains three parameters which give a weight according to the stockholders' and company's desires. The additional terms as well as the parameters associated with them might be considered as "utility terms."

In Chapter IV, a parametric analysis of the two models is performed using two numerical examples. In order to judge the performance of the proposed models, three criteria are used: the terminal wealth, the average dividend, and the standard deviation of the stream of dividend payments. The responses of these three criteria given the levels of parameter values is studied, and conclusions are drawn. Finally, in Chapter V, there is presented a summary of the conclusions obtained from the study of the models' behavior. The drawbacks found by the author are discussed, and a group of recommendations are listed.

CHAPTER II

LITERATURE SURVEY

In the last 25 years many researchers have developed different approaches for capital budgeting models regarding dividend policy. Most of these approaches use linear programming models, though there are also some nonlinear models. First, in order to obtain some background in the area of Mathematical Programming in Capital Budgeting, it is necessary to review some of the more important contributions to the field.

The Basic Horizon Model

Among the better known models, the Basic Horizon Model of Weingartner (19) has special importance because of the way he maximizes the terminal wealth of the company. Basically, his model can be stated as follows:

Maximize:

$$\sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T \tag{2.1}$$

Subject to:

$$\sum_{j=1}^{n} a_{1j} x_j + v_1 - w_1 \leq M_1 \tag{2.2}$$

$$\sum_{j=1}^{n} a_{tj}x_j - (1+r)v_{t-1} + v_t + (1+r)w_{t-1} - w_t \leq M_t \qquad t = 2,3,\ldots, T \tag{2.3}$$

$$0 \leq x_j \leq 1 \qquad j = 1,2,\ldots, n \tag{2.4}$$

$$v_t,\ w_t \geq 0 \qquad t = 1,2,\ldots, T \tag{2.5}$$

where $\hat{a}_j$ is the present worth computed at the horizon year T of all the cashflows of the project j occurring after year T.

a_{tj} is the cashflow associated with project j occurring within the horizon period.

M_t is the cash available from other sources for year t.

r is the interest rate for borrowing and lending.

T is the horizon year.

v_t is the amount to be lent at year t.

w_t is the amount to be borrowed at year t.

x_j is the proportion of project j which is accepted.

It can be shown using the linear programming dual of the model, that maximizing the terminal wealth subject to the constraints is equivalent to maximizing the present worth at the horizon year of all the cashflows associated with the accepted projects, where project present worth is computed at interest rate r. The Basic Horizon Model does not have any constrains about the amount of money to be lent and borrowed, nor does it include any dividend variables. The model is important because it demonstrates a fundamental relationship between project acceptance using interest rate and project evaluation using dual variables.

Weingartner's Dividend Model

Weingartner (20) also proposes a model including a dividend policy, as follows:

Maximize:

$$D_T \tag{2.6}$$

Subject to:

$$-\sum_{j=1}^{n} a_{1j}x_j + v_1 - w_1 + D_1 \leq M_1 \tag{2.7}$$

$$-\sum_{j=1}^{n} a_{tj}x_j - (1+r_{t-1})v_{t-1} + v_t + (1+r'_{t-1})w_{t-1}$$
$$-w_t + D_t \leq M_t \qquad t = 2,3,\ldots, T \tag{2.8}$$

$$D_1 \geq D_{min} \tag{2.9}$$

$$D_t \geq D_{t-1} \qquad t = 2,3,\ldots, T \tag{2.10}$$

$$r\{\sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T + M\} \geq D_T \tag{2.11}$$

$$0 \leq x_j \leq 1 \qquad j = 1,2,\ldots, n \tag{2.12}$$

$$v_t,\ w_t \geq 0 \qquad t = 1,2,\ldots,\ T \tag{2.13}$$

where:

$$M = \sum_{t=T+1}^{\infty} A_t(1+r)^{T-t} \tag{2.14}$$

(residual value of physical assets which existed at the start of the planning period)

r_{t-1} is the lending rate for t-1.

r'_{t-1} is the borrowing rate for t-1.

This model contains a number of interesting features. Constraint (2.11) relates the terminal wealth to D_T with the discount rate r. It establishes that all the post-horizon dividend payments should be greater than or equal to D_T assuming that all the terminal wealth is used to pay dividends. The term between braces in (2.11) represents the terminal wealth. It contains the net amount of financial assets $(v_T - w_T)$ and the physical assets $(\Sigma\hat{a}_j x_j + M)$. The latter consists of the residual values (at the horizon) of the assets resulting from the investments undertaken and those which existed at the start. Cash flows of the latter are denoted by A_t, and their residual value as of the horizon is denoted by M.

According to Weingartner, by maximizing the dividend during year T, subject to the constraint that this level of dividends can be maintained, the model also maximizes the rate of growth of dividends. Weingartner points out that the model shown above has two drawbacks. First, maximization of the growth rate of dividends up to the horizon may produce a stream of dividends which is small over a large portion of the time to the horizon, but rises sharply at the end. Second, it yields a single solution rather than a set of alternatives from which the optimal pattern of dividends, within the policy limits, can be chosen.

In order to meet those objections, constraint (2.10) is replaced by a minimum growth rate in dividends K:

$$D_t \geq KD_{t-1} \tag{2.15}$$

The constant K is determined by the company's decision maker as an index of the dividend policy which he would like to have.

Given the structure of the constraints of the Weingartner's dividend model, it does not allow the dividends to fluctuate up and down. So, the company always has to pay dividends greater than or equal to those during the last year. Under these circumstances, even though the earnings may be high in one year, the company will not necessarily pay high dividends because if it does, it will have to pay dividends as high as those in the following years. This might result in solutions with low dividend payments which are not attractive for stockholders.

Bernhard's Dividend Model

Bernhard's general model (4) has also special importance because he was one of the first ones who introduced in an explicit way the "Terminal Wealth" concept. He argues that the objective function should be a function of the dividend payments within the horizon, as well as of the terminal wealth:

Maximize:

$$f(D_1, D_2, \ldots, D_T, G) \qquad (2.16)$$

where D_t is the dividend payment in year t.

G is the terminal wealth

$$G = M + \sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T + (c_T w_T + C_T) \qquad (2.17)$$

where M is the present worth of post-T cash flows from outside projects and other outside sources.

$c_T w_T + C_T$ is the "liquidity restriction" for year T. These "liquidity restrictions" are included in each one of the cash balance restrictions, and they represent an amount of money to be carried from t to t+1.

The terminal wealth is constrained to be positive by an inequality of the type:

$$G \geq K + g(D_1, D_2, \ldots, D_T) \qquad (2.17)$$

where K is a nonnegative constant.

The objective function (2.16) maximizes f. But this function as well as g in (2.17), are not specified by Bernhard. By constraint (2.17), the company is enabled to pay post-T dividends that are related to those paid within the horizon.

In addition to the cash balance and terminal wealth restrictions, other special constraints are specified by Bernhard, as the following:

a) Group payback restriction:

$$-\sum_{j=1}^{n}\sum_{t=1}^{t'} a_{tj}x_j \leq 0 \qquad j = 1,\ldots, n \tag{2.18}$$

For some integer t', such that $1 \leq t' \leq T$

b) Scarce material restriction:

$$\sum_{j=1}^{n} k_j x_j \leq K \qquad j = 1,\ldots, n \tag{2.19}$$

c) Upper bound on borrowing restrictions

$$w_t \leq B_t \qquad t = 1,\ldots., T \tag{2.20}$$

Even though Bernhard criticizes Weingartner because of the way he specified his objective function (19), Bernhard himself never specified a form for his objective function (4). However, he mentions that the suggestions regarding the objective function formulation by Baumol and Quandt (2) "were another source in the formulation of that (his) model." It can be interpreted that he accepts or at least does not disagree with their formulation of an objective function, which has the following form:

$$f(D_1, D_2, \ldots, D_T, G) = \sum_{t=1}^{T} u_t D_t \qquad u_t \geq 0 \qquad (2.21)$$

There are many people who accept this type of objective function. One of them is Manne (13), who showed that a nonlinear utility objective function works well assuming concavity. He also showed that the linear utility objective function (2.21), yields solutions in which dividend payments are not allowed in years in which the company makes an investment, assuming that all the projects are of deterministic type known as "point input-stream output." Due to this fact, he points out the desirability of using nonlinear utility objective functions.

Unfortunately, as Weingartner (20) points out, an objective function like (2.21) has a lack of realism:

a) It assumes an identity between the firm and the owner-enterpreneur.

b) It requires an assignment of the utility index in advance of information about the withdrawal possibilities.

c) It makes the utility of withdrawal in one period independent of the amount available in another.

In addition to these items, the objective function does not consider in any way the company's terminal wealth.

In esscence, Bernhard's model does not consider any dividend policy at all. The dividend payments within the horizon may fluctuate from zero up to the maximum amount of money which the company can obtain in a specific year, based on the budget constraint, the borrowing constraint, and the cash inflows from projects.

The Hamilton and Moses Model

Hamilton and Moses (9) take a different approach to the problem of capital budgeting. Their model, with a wider point of view, takes into account sale and purchase of stock, as well as considerations of earnings, long-term and short-term debt, preferred stock, dividends, and several financial ratios. They argue that the most reasonable surrogate measure of corporate performance is the earnings per share (EPS). Their objective is, of course, to select a good measurement of the value of the corporation to its stockholders in order to maximize it, and given that the dividends valuation models imply the use of an "appropriate cost of capital rate," which is a matter of considerable debate, they prefer to use EPS. So, the objective function they present terms out to be:

Maximize:

$$EPS = \sum_{t=1}^{T} E_t/S_0 \qquad (2.22)$$

where: E_t are the earnings during year t.

S_0 is the constant number of common shares outstanding at t=0.

T is the planning period.

This objective function can be rewritten to take into account the logical expansions and contractions of the stock base S_0, which might be expected during the planning period. It should be noted from the objective function that there is no consideration of the corporate terminal wealth, and of any post-horizon cash flow either, but the series of earnings per share is truncated at the horizon.

Even though the earnings at t might fluctuate up and down, they are constrained by an upper and lower bounds set on earnings at t-1. This, in fact, represents a stable growth policy in EPS within the horizon. But here again, nothing is said about the post-horizon growth. Hamilton and Moses report in their paper the necessity of Mixed Integer Programming given that some of the variables in their model were zero-one variables, and some of the continuous type.

The Carleton Model

Another interesting model is given by Carleton (5), who uses accounting tools in formulating it. Like Hamilton and Moses, he argues that the main objective of a company is to maximize its value to its stockholders. Carleton, however, maximizes the share price and his objective function has the following form:

Maximize:

$$\frac{P_0}{N_0} \tag{2.23}$$

where: P_0 is the aggregate market value at the beginning of the planning period.

N_0 is the number of shares of common stock outstanding at the beginning of the planning period.

The share price P_0/N_0 is defined by Carleton as follows:

$$\frac{P_0}{N_0} = \frac{D_0}{N_0} + \frac{D_1}{N_1(1+k_1)} + \cdots + \frac{D_{j'}}{N_{j'}(1+k_1)\ldots(1+k_{j'})}$$
$$+ \frac{D_{j'+1} - \Delta\bar{E}^N_{j'+1}}{N_{j'}(1+k_1)\ldots(1+k_{j'+1})} + \frac{D_{j'+2} - \Delta\bar{E}^N_{j'+2}}{N_{j'}(1+k_1)\ldots(1+k_{j'+2})} \tag{2.24}$$
$$+ \frac{D_{n-1} - \Delta\bar{E}^N_{n-1}}{N_{j'}(1+k_1)\ldots(1+k_{n-1})} + \frac{P_n - \Delta\bar{E}^N_n}{N_{j'}(1+k_1)\ldots(1+k_n)}$$

where: D_t are the aggregate dividend payments of t

N_t is the number of shares of common stock outstanding at t.

k_t is the rate of return "required by the stock market" of the firm between periods t and t+1.

$\Delta\bar{E}_t^N$ is the net (always positive) increment of equity after stock flotation costs.

P_n is the aggregate market value at the end of the planning period.

In order to take into account the post-horizon dividend cash flows, Carleton assumes that after that moment, the company enters a steady state, in which there is a constant post-horizon dividends growth rate $\bar{g}$. So, the market value at the horizon is given by the well known formula:

$$P_n = \frac{D_n}{k_n - \bar{g}} \tag{2.25}$$

In addition to the objective function stated above, Carleton established some institutional and corporate policy constraints. Some of them are related to the debt equity ratio, investment requirements, and other things, but the most interesting ones are the constraints imposed on the dividends and earnings available to the stockholders:

$$D_t \geq c_1 D_{t+1} \tag{2.26}$$

$$AFC_t \geq c_2 AFC_{t+1} \tag{2.27}$$

where: AFC_t are the earnings available to the stockholders.

D_t the dividend payments at t and

c_1, c_2 are constants such that $1 \leq c_1 \leq c_2$

Carleton says that these restrictions on D_t and AFC_t allow the corporate planner to offset indirectly omission of the dynamics of expectations formuation. Also, a deterministic model which allow dividends per share or earnings per share to fluctuate freely would be correct given his assumptions but unrealistic, because there is information content in a dividend or earnings per share, which if uncontrolled, could lead to nonviable market expectations and to changes in required rates of return. However, he did not explain what information he referred to, and how the "nonviable market expectations" might be determined.

Beside the individual restrictions imposed on dividends and earnings, Carleton sets upper and lower bounds on the payout ratios:

$$D_t \geq c_3 \ AFC_t \tag{2.28}$$

$$D_t \leq c_4 \ AFC_t \tag{2.29}$$

where: c_3 and c_4 are nonnegative constants.

The Miller-Modigliani Position

One of the greatest theoretical contributions to the Financial Management field in the last 20 years has been the theory of Miller-Modigliani (15), who argue that the firm's current policy has no effect on the current price of its shares. Given this, Miller-Modigliani have a position against the traditional dividends valuation models, which, as used by Carleton, maximize the share price selecting an appropriate dividend policy within the horizon.

The Miller-Modigliani position (MM) has caused an unending debate among theoreticians in the finance field.

Among the people who support MM are Walter (21) and Higgins (10). On the contrary, people opposed to them are Gordon (8), Robichek and Myers (17), Chen (6), Van Horne and McDonald (14) and others.

The MM model, in order to be valid, needs several assumptions which can be summarized as follows:

1) There is assumed a "perfect capital market," which implies that:
   - no buyer or seller of securities is large enough for his transactions to have an appreciable impact on the then ruling price.
   - all traders have equal and costless access to information about the ruling price and about all other relevant characteristics of shares.
   - no brokerage fees, transfer taxes or other transaction costs are incurred when securities are bought, sold, or issued, and there are no tax differentials either between distributed and undistributed profits or between dividends and capital gains.
2) There is assumed "rational behavior" which means that investors always prefer more wealth to less, and are indifferent as to whether a given increment to their wealth takes the form of cash flows or an increase in the market value of their holdings of shares.
3) There is assumed "perfect certainty," which implies complete assurance on the part of every investor as to the

future investment program and the future profits of every corporation.

The irrevelance of dividends in the share price is not accepted by many people, some of them mentioned above. The arguments presented by those against the MM position are supported by the logic and by the observation of the real behavior of an investor. Those arguments according to Van Horne (18) can be summarized as follows:

1) Payment of current dividends resolves uncertainty in the minds of investors and therefore, an investor is not indifferent between dividends and capital gains, and is willing to pay a higher price for the stock that offers a greater current dividend.
2) The dividends have an information content about the firm's profitability, and this used by an investor in the stock purchase decision and dividends therefore affect the stock price.
3) The investors are not indifferent to receiving dividends or capital gains. They have preferences for one of them and pay for it.

In a survey made by Harkins and Walsh (23), using information from 166 members of a panel of senior financial executives who responded to an inquiry concerning their company's dividend practices, there are reported several useful conclusions:

1) The items of "continuity or regularity of dividend payments"

and "stability of rate per share" are included in 26 percent and 16 percent of the responses, respectively, to a question concerning which factors were important for dividend decisions. From this, it seems that the idea of paying a continuous and steady stream of dividends represents a major concern for many managers.

2) About 50 percent of the respondents say that despite the logical reasons given by theory (capital gain benefit stockholders), most stockholders <u>expect</u> to receive current income from their investments and, therefore, boards of directors cannot afford to ignore their demands. Basically, this response represents empirical evidence of the preference of stockholders, at least in some cases, for current and steady dividends. We say in some cases, because the stockholders can be differentiated into groups. For example, owners of closely held corporations prefer to forego current dividend income so that all earnings can be plowed back into expanding their business; elderly retired persons, who need all the income they can obtain, favor generous and steady current dividend payouts, etc.

Regarding the present work, the typically observed behavior of an investor is assumed, so the MM position is not considered in our study.

Summary

Several points of view which are considered important were discussed in this chapter. They represent valuable tools which are used to help formulate the models suggested by the author in the next chapter.

CHAPTER III

MODEL FORMULATION AND ANALYSIS

In Chapter II, we reviewed some classical capital budgeting models which were developed using quite different approaches. Some of them have drawbacks which, in some cases, are recognized by their own authors. As a matter of face, the "steadiness" of dividend payments is not considered at all in any of the cases. This "steadiness" is regarded by the author as a very important factor for any stockholder, and should be considered somehow in the structure of a capital budgeting model. In addition, it is desired to maintain some relationship between the terminal wealth and the average dividend paid during the planning period. So, it is intended in this chapter to formulate two models, one linear and one nonlinear for the purpose of obtaining solutions which satisfy these objectives.

Models Presentation

The constraints of the models suggested by the author are similar to those presented by Weingartner (19) and Bernhard (4). The main differences take place in the objective function, to which several additional terms have been added. The purpose of these terms is to diminish the variability of dividends and keep them near a desired level.

Different from Weingartner's model, the objective functions of the models presented here do not maximize just the terminal wealth (at the horizon). What they attempt to do is to maximize the terminal wealth plus the stream of dividends discounted to the beginning of the planning period. The idea of discounting the dividend payments and the terminal wealth to $t = 0$ recognizes the importance of both of these items; similar results could have been obtained if they would have been discounted to the horizon. The discount rate used in the objective function of both models is the stockholders' discount rate, and it is assumed throughout that it is known in advance.

A term added to the objective function in both models measures, as mentioned above, the variability of dividend payments and it might be regarded as a "utility term." Another objective function term represents the difference between a measurement of dividends and the terminal wealth. The higher the value of this term, the lower the value of the objective function.

The way in which weights are assigned to these "utility terms" is by using three parameters c_1, c_2, and c_3. These parameters control the trade-off between dividend payments and terminal wealth, and should reflect the views of the stockholders and management. A parametric analysis should be made to determine the best set of projects that the company should select, and the way it is going to finance them.

Linear Model

Our proposed linear model has n + 3(T+1) variables, n + 2 (T+2) constraints, and the following structure:

Maximize:

$$\sum_{t=0}^{T} \frac{D_t}{(1+k)^t} + [\sum_{j=1}^{n} \hat{a}x_j + v_T - w_T]/(1+k)^T - c_1 [\frac{\sum_{t=0}^{T} D_t(t - \frac{T}{2})}{(T+1)(\frac{T(2T+1)}{6}) - \frac{T^2}{4})}]$$

$$-c_2[(\sum_{j=1}^{n} \hat{a}_j \; x_j + v_T - w_T)/(1+k)^T c_3 - \frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^t}] \tag{3.1}$$

Subject:

$$[f_0] \qquad -\sum_{j=1}^{n} a_{0j}x_j + v_0 - w_0 + D_0 \leq M_0 \tag{3.2}$$

$$[f_t] \qquad -\sum_{j=1}^{n} a_{tj}x_j + v_t - w_t + D_t - (1+r_{\ell t})v_{t-1} + (1+r_{bt})w_{t-1} \leq M_t \qquad t = 1,2,\ldots, T \tag{3.3}$$

$$[g] \qquad [\frac{\sum_{t=0}^{T} D_t(t - \frac{T}{2})}{(T+1)(\frac{T(2T+1)}{6}) - \frac{T^2}{4})}] \geq 0 \tag{3.4}$$

$$[h_t] \qquad D_t \geq D_{min} \qquad t = 0,1,\ldots,T \tag{3.5}$$

[i] $$\left[\sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T\right] / c_3 (1+k)^T - \frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^t} \geq 0 \qquad (3.6)$$

$$v_t,\ w_t \geq 0 \qquad t = 0,1,\ldots,T \qquad (3.7)$$

[ℓ_j] $$0 \leq x_j \leq 1 \qquad j = 1,2,\ldots,n \qquad (3.8)$$

where: a_{tj} is the cash flow of project j at year t

$\hat{a}_j$ is the present worth at the horizon of all the post-horizon cash flows of project j discounted at k.

c_1, c_2, c_3 are the nonnegative parameters which assign weights to the utility terms.

D_t is the dividend payment at year t.

D_{min} is the minimum annual dividend payment within the horizon.

k is the stockholders' discount rate

M_t is the money available from other sources for year t.

r_{bt} is the borrowing rate of the money borrowed at t-1 and paid at t.

$r_{\ell t}$ is the lending rate of the money lent at t-1 and paid at t.

T is the horizon year.

v_t is the money lent at t.

w_t is the money borrowed at t.

x_j is the decision variable for project j.

The corresponding dual variables have been placed on the left of each constraint. These will be used in the optimality analysis shown in the following pages.

Given the objective function above, let G be the company's terminal wealth defined by Bernhard (4):

$$G = \sum_{j=1}^{n} \hat{a}_j x_j + v_t - w_T + H \qquad (3.9)$$

where H is the time T present worth of post-horizon cash flows from outside projects and other outside sources. H in our case is just a constant which for our purposes can be dropped without affecting the final result. So, terminal wealth can be redefined:

$$G' = \sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T \qquad (3.10)$$

The first two elements of the objective function (3.1) represent the time zero present worth of the dividend payments within the horizon, and the time zero present worth of the terminal wealth G′. With these two elements in the objective function, the interests of both management and stockholders are recognized.

The third term in (3.1) reflects the variability of the dividend payments. This term tries to minimize the slope b_1 of the dividends, where b_1, the best unbiased estimator, is defined as follows:

$$b_1 = \frac{\sum_{t=0}^{T} (tD_T) - [\sum_{t=0}^{T} t \sum_{t=0}^{T} D_t]/(T+1)}{\sum_{t=0}^{T} t^2 - [\sum_{t=0}^{T} t]^2/(T+1)} \tag{3.11}$$

Note that given a horizon of T years, we have T+1 dividend payments assuming that also in t=0 the company pays dividends. With that in mind, and knowing that:

$$\sum_{t=0}^{T} t = \frac{T(T+1)}{2} \tag{3.12}$$

$$\sum_{t=0}^{T} t^2 = \frac{T(T+1)(2T+1)}{6} \tag{3.13}$$

We can show that the slope b_1 becomes:

$$b_1 = \frac{\sum_{t=0}^{T} D_t(t - \frac{T}{2})}{(T+1)[\frac{T(2T+1)}{6} - \frac{T^2}{4}]} \tag{3.14}$$

The parameter c_1 represents the weight that the slope b_1 has in the objective function. In theory, the optimum value that b_1 should have would be zero, but this does not mean that all the dividends are equal to each other. This represents, as we will show later, one of the drawbacks of our linear model. Given that no stockholder would like to have a decreasing trend in his dividend payments, b_1 is restricted to be nonnegative in (3.4).

The fourth and last term in (3.1) represents the difference between the terminal wealth and the stream of dividend payments. In order to do that, the time zero terminal wealth G' divided by the parameter c_3 is compared with the average of the time-zero present worths of the dividends. In addition, a weight given by the parameter c_2 is assigned to the difference. Let us explain why this term is written this way.

If we take the ratio of the average dividend to terminal wealth, we obtain:

$$\frac{\frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}}{\frac{G'}{c_3(1+k)^T}} = \left(\frac{c_3}{T+1}\right)\left[\frac{\sum_{t=0}^{T} D_t(1+k)^{T-t}}{G'}\right] \tag{3.15}$$

Note that the second factor of the right hand side is the ratio of the time T future worth of dividends to the time T terminal wealth and may be regarded as some sort of pay-out for the horizon. (Pay-out is defined as the ratio of dividends to earnings.) In other words, the "horizon pay-out" represents the effective percentage of the company's wealth which is distributed among the stockholders. At the optimum, we would like to have the equation (3.15) equal to one, and if we assume so, we can say that:

$$\frac{\sum_{t=0}^{T} D_t(1+k)^{T-t}}{G'} = \frac{T+1}{c_3} \tag{3.16}$$

The above means that the ratio $(T+1)/c_3$ represents, at the optimum, the "horizon pay-out" ratio also. As can be seen, the "horizon pay-out" depends upon the length of the horizon itself, as well as the value of the parameter c_3, which should be established according to the company's dividend policy.

In addition, a parameter c_2 gives the weight to the whole difference term. What c_2 measures is the importance of that term in the objective function. If we let the difference between G' and dividends have positive or negative sign, a solution in which the company pays too much in dividends might be obtained, with the consequence that very attractive projects may be rejected in order to use the money to pay dividends. So, to avoid that situation the difference between G' and dividends is constrained to be nonnegative in (3.6).

Another feature of dividend policy which any company should adopt is contained in (3.5), in which the dividend payments within the horizon are constrained to be greater than or equal to D_{min}. This minimum amount of dividends is set according to the historical data which the company has for the time being in business, and it represents the minimum return which stockholders are willing to receive.

The other constraints of the model are the budget constraints, the nonnegativity requirements for v_t, w_t, and x_j, and those which constrain x_j to be less or equal to one.

All of these are basically similar to those given by Weingartner and Bernhard, and need no further explanation.

Optimality Analysis of the Linear Model

The dual problem of the linear model can be established without difficulty (3):

Minimize:

$$\sum_{t=0}^{T} M_t f_t + \sum_{t=0}^{T} D_{min}\, h_t + \sum_{j=1}^{n} \ell_j \tag{3.17}$$

Subject to:

$$[D_t] \quad f_t + [\frac{t - (\frac{T}{2})}{C}]g + h_t - \frac{i}{(T+1)(1+k)^t} \geq$$

$$\{\frac{1}{(1+k)^t}[1 + \frac{c_2}{T+1}] - \frac{c_1}{C}(t - \frac{T}{2})\} \tag{3.18}$$

$$t = 0,1,\ldots,T$$

$$-\sum_{t=0}^{T} a_{tj} f_t + \frac{\hat{a}_j i}{c_3(1+k)^T} + \ell_j \geq \frac{\hat{a}_j}{(1+k)^T}[1 - \frac{c_2}{c_3}]$$

$$j = 1,2,\ldots,n \tag{3.19}$$

$$[v_t] \quad f_t - (1+r_{\ell t+1})f_{t+1} \geq 0 \qquad t = 0,1,\ldots,T-1 \tag{3.20}$$

$$[v_t] \quad f_T + \frac{i}{c_3(1+k)^T} \geq \frac{1}{(1+k)^T}[1 - \frac{c_2}{c_3}] \tag{3.21}$$

$$[w_T] \qquad -f_t + (1 + r_{bt+1})f_{t+1} \geq 0 \qquad t = 0,1,\ldots,T-1 \tag{3.22}$$

$$[w_T] \qquad -f_T - \frac{i}{c_3(1+k)^T} \geq - \frac{1}{(1+k)^T}\left[1 - \frac{c_2}{c_3}\right] \tag{3.23}$$

$$f_t,\ \ell_j \geq 0 \qquad h_t,\ i,\ g \leq 0 \tag{3.24}$$

where

$$C = (T+1)\left[\frac{T(2T+1)}{6} - \frac{T^2}{4}\right] \tag{3.25}$$

The corresponding primal variables have been put in brackets at the left hand side of each constraint.

Dual restrictions (3.20) and (3.21) are derived from lending activities, while (3.22) and (3.23) come from borrowing activities. The two conditions (3.20) and (3.22) written together are:

$$(1 + r_{\ell t+1}) \leq \frac{f_t}{f_{t+1}} \leq (1 + r_{bt+1}) \tag{3.26}$$

which imply the logical conclusion and necessary condition:

$$r_{bt} \geq r_{\ell t} \qquad \text{for any } t \tag{3.27}$$

We say necessary condition because if the inequality had the opposite direction, it would imply that the company always would borrow money for lending it.

Combining (3.21) and (3.23), we obtain:

$$f_T + \frac{i}{c_3(1+k)^T} = \frac{1}{(1+k)^T}\left(1 - \frac{c_2}{c_3}\right) \tag{3.28}$$

Making some transformations, it may be easily shown that:

$$i = c_3[1 - (1+k)^T f_T] - c_2 \tag{3.29}$$

Knowing from (3.24) that i should be nonpositive, (3.29) may be rewritten in the following way:

$$c_3[1 - (1+k)^T f_T] - c_2 \leq 0 \tag{3.30}$$

or

$$f_T \geq \left(1 - \frac{c_2}{c_3}\right)(1+k)^{-T} \tag{3.31}$$

Again from (3.24), we know that f_T should be nonnegative which implies that:

$$\frac{c_2}{c_3} \leq 1 \tag{3.32}$$

or

$$c_2 \leq c_3 \tag{3.33}$$

Note that if $c_2 \geq c_3$, the terminal wealth, instead of increasing the value of the objective function, would decrease it.

Restriction (3.33) might be easily interpreted. The weight c_2 represents the "importance" that the decision-maker assigns to the objective of having a "horizon pay-out" equal to $(T+1)/c_3$. This is clearly seen if we take an extreme case. Let $c_3 = c_2$. The terminal wealth would not be present in the objective function because it is cancelled out by the ratio c_2/c_3. On the other hand, equation (3.29) may be rewritten as:

$$i = -(1+k)^T f_T \tag{3.34}$$

From restriction (3.26), we know that f_T should not be zero, and hence neither should i. So, using the Kuhn-Tucker conditions, constraint (3.6) becomes an equation at the optimum. This means that the dividend payments are set to such values that (3.6) is satisfied as an equation, and so the "horizon pay-out" is exactly $(T+1)/c_3$. So, the higher the importance of getting a specific "horizon pay-out," the higher the value of c_2.

Returning to the general case, we may note that when a project j is accepted ($x_j>0$), constraint (3.19) becomes an equality, and if we substitute the value for i from (3.29) into (3.19), it may be shown that:

$$\ell_j = \hat{a}_j f_T + \sum_{t=0}^{T} a_{tj} f_t \tag{3.35}$$

We may note that $\ell_j=0$ for those projects which are rejected or partially accepted ($0\leq x_j\leq 1$), so the last term in (3.17) is a function of the cash flows associated with the fully accepted projects:

$$\sum_{j=1}^{n} \ell_j = \sum_{j\varepsilon S'} {}_j = f_T \sum_{j\varepsilon S'} \hat{a}_j + \sum_{j\varepsilon S'} \sum_{t=0}^{T} a_{tj} f_t \qquad (3.36)$$

where S′ is the set of fully accepted projects.

From the dividend policy established by our model, D_t should always be greater than or equal to D_{min}. When it happens that D_t is strictly greater, the slack of the dual constraint (3.18) is zero, and this constraint may be rewritten as an equation:

$$f_t + [\frac{t - \frac{T}{2}}{C}]g + h_t - \frac{i}{(T+1)(1+k)^t}$$

$$= \frac{1}{(1+k)^t}[1 + \frac{c_2}{T+1}] - \frac{c_1}{C}(t - \frac{T}{2}) \qquad (3.37)$$

$$t = 0,1,\ldots, T$$

$$D_t > D_{min}$$

On the other hand, if the value of i from (3.29) is substituted in (3.18), it is easy to show that:

$$f_t + h_t + \frac{1}{(1+k)^t}[\frac{c_3(1+k)^T}{T+1} - \frac{c_3}{T+1} - 1]$$

$$\geq [\frac{T}{2} - t][\frac{c_1 + g}{C}] \qquad (3.38)$$

$$t = 0,1,\ldots, T$$

or

$$h_t \geq \frac{1}{(1+k)^t} [1 - \frac{c_3(1+k)^T}{T+1} f_T + \frac{c_3}{T+1}] - f_t + [\frac{T}{2} - t][\frac{c_1+g}{C}] \tag{3.39}$$

$$t = 0,1,\ldots, T$$

If we add up all the dual variables h_t for the planning period, (3.39) becomes:

$$\sum_{t=0}^{T} h_t \geq \sum_{t=0}^{T} \frac{1}{(1+k)^t} [1 - \frac{c_3(1+k)^T}{T+1} f_t + \frac{c_3}{T+1}] - \sum_{t=0}^{T} f_t$$

$$+ \sum_{t=0}^{T} (\frac{T}{2} - t) (\frac{c_1 + g}{C}) \tag{3.40}$$

$$t = 0,1,\ldots, T$$

but we know that:

$$\sum_{t=0}^{T} (\frac{T}{2} - t) = \frac{T(T+1)}{2} - \frac{T(T+1)}{2} = 0 \tag{3.41}$$

$$\sum_{t=0}^{T} (1+k)^{-t} = \frac{(1+k)^T - 1}{k(1+k)^T} = K \tag{3.42}$$

Then (3.39) may be rewritten in a simpler way:

$$\sum_{t=0}^{T} h_t \geq K + K[\frac{c_3}{T+1}][1-(1+k)^T f_T] - \sum_{t=0}^{T} f_t \tag{3.43}$$

Equation (3.36) and constraint (3.43) may be substituted in the dual objective function. But given that (3.43) is an inequality and the dual problem is a minimization problem, the right hand side of (3.43) produces a lower bound in the objective function. It is not difficult to show that the lower bound is:

$$\sum_{t=0}^{T} [M_t + \sum_{j\varepsilon S'} a_{tj} - D_{min}] f_t + [\sum_{j\varepsilon S'} \hat{a}_j - KD_{min}(\frac{c_3}{T+1})(1+k)^T] f_T$$

$$+ KD_{min} \qquad (3.44)$$

Note that the last term KD_{min} is just a constant which does not affect the optimal solution in the minimization process, and it can be dropped.

It may be noted that (3.44) is not a function of c_1 and c_2. That means that the optimal solution, when it gets closer to its lower bound, becomes independent of these two parameters. This is another drawback of the linear model.

The above may also be seen if (3.1) is rewritten as follows:

Maximize:

$$\sum_{t=0}^{T} \{\frac{1}{(1+k)^t}[1 + \frac{c_2}{T+1}] - \frac{c_1(t-\frac{T}{2})}{C}\} D_t + \frac{G'}{(1+k)^T} [1 - \frac{c_2}{c_3}] \qquad (3.45)$$

All the terms which compose the slope as well as the difference term in the original primal objective function (3.1), have been factored out in such a way that they represent no longer what they originally did.

Range of Parameters of the Linear Model

Parameter c_1, as established in the primal of the linear model is the weight that the slope b_1 has in the objective function. It should work against the dividend variability. In order to see what its range is, the objective function rewritten as in (3.45) should be analyzed.

The coefficient of D_t, as well as the sign of the coefficient, give to D_t the necessary weight to become greater than or just equal to D_{min}. So, the value of that coefficient should not be much different from $1/(1+k)^t$, otherwise an undesireable stream of dividends might be obtained. As a general rule proposed by the author and obtained from a series of runs and tests, c_1 should be restricted as follows:

$$\frac{c_1}{C} \leq 1 \tag{3.46}$$

or

$$c_1 \leq (T+1)[\frac{T(2T+1)}{6}] - \frac{T^2}{4}] \tag{3.47}$$

A heuristic restriction that the author found worked well, is:

$$c_2 \leq T+1 \tag{3.48}$$

It must be recalled that c_2 is also constrained by (3.33). Anyway, small deviations from that established by (3.47) and

(3.48) might work depending upon the set of projects for analysis as well as the discount rate k.

The criteria for selecting the value of c_3 was mentioned before, when we discussed the "horizon pay-out." This ratio, if well interpreted, may not be very difficult to be established.

Nonlinear Model

Our proposed nonlinear model has n + 3(T+1) variables, n + 2(T+1) constraints and the following structure:

Maximize:

$$\left[\sum_{t=0}^{T} \frac{D_t}{(1+k)^t}\right] + \left[\sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T\right]/(1+k)^T$$

$$- c_1 \sum_{t=1}^{T} (D_t - D_{t-1})^2 - c_2\{\left[\sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T\right]/c_3(1+k)^T \tag{3.49}$$

$$- \frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^t}\}^2$$

Subject to:

$$[(u_M)_0] \quad -\sum_{j=1}^{n} a_{0j} x_j + v_0 - w_0 + D_0 \leq M_0 \tag{3.50}$$

$$[(u_M)_t] \quad -\sum_{j=1}^{n} a_{0j} x_j + v_t - w_t + D_t - (1+r_{\ell t}) v_{t-1}$$

$$\tag{3.51}$$

$$+ (1+r_{bt}) w_{r-1} \leq M_t \qquad t = 1,2,\ldots,T$$

$$[(u_D)_t] \quad D_t \geq D_{min} \qquad t = 0,1,\ldots, T \tag{3.52}$$

$$[(u_x)_j] \quad 0 \leq x_j \leq 1 \qquad j = 1,2,\ldots, n \tag{3.53}$$

$$v_t,\ w_t,\ D_t \geq 0 \qquad t = 0,1,\ldots, T \tag{3.54}$$

All the decision variables as well as the terminal wealth are defined in the same way as they were in the linear model. However, the parameters c_1, c_2, and c_3 have slightly different interpretation as we will see later.

Here again, the first two terms of (3.49) represent the time zero present worth of the dividend payments within the horizon, and the time zero present worth of the terminal wealth G'. As commented earlier, these two elements reflect the interests of both management and stockholders.

The third term has the effect of reducing variability of dividends. With the aid of nonlinear programming, the quadratic differences of dividend payments are minimized according to a weight c_1 assigned to them. Note that given T+1 dividend payments, there are T quadratic differences. The parameter c_1 has a value according to the importance given by the decision-maker to dividend variability. Some rules for finding its value are given later.

The fourth term of the nonlinear objective function represents the difference between terminal wealth and dividend payments. This difference is established as in the linear model, but here the difference is squared. The parameter c_3 divides the terminal wealth G', and this ratio is compared with the average time zero present worth of the dividend payments as before. The ratio $(T+1)/c_3$ is also here interpreted as a "horizon pay-out" as before. The parameter c_2 gives the weight to the quadratic difference.

Because of the use of quadratic terms in the objective function, there is no need to constrain the difference between dividends and the terminal wealth to be positive, so the nonlinear model becomes simpler than the linear case as far as constraints is concerned. The set of constraints of the nonlinear model are exactly the same as those for the linear model after dropping (3.4) and (3.6). This means that the set of constraint is reduced to: budget, dividend policy, and nonnegativity constraints. As in the linear model, the dual variables for the nonlinear case are put at the left hand side of the primal constraints, between brackets.

Basically, the present model attempts to do the same things as the linear one, that is, to plan the investments for the horizon taking care of obtaining a steady and fair stream of dividends. However, the approaches for obtaining it are quite different. It may be anticipated that the nonlinear model obtains better results than the linear case.

Optimality Analysis of the Nonlinear Model

The formulation of the dual for the present model is not as easy as in the first case. First, it is convenient to write down the standard form of the general quadratic model (23):

Maximize:

$$\underline{q}^t\underline{z} + \underline{z}^t Q \underline{z} \tag{3.55}$$

Subject to:

$$A\underline{z} \leq b \tag{3.56}$$

$$z \geq 0 \tag{3.57}$$

where: q is an r-component row vector and contains the coefficients of the linear terms z_i of the objective function.

z is an r-component column vector of the primal variables.

Q is an rxr matrix and contains elements c_{ij} such that:

$$c_{ij} = \begin{cases} \text{coefficient of } z_i z_j \text{ in the objective function if } i = j. \\ \\ 1/2 \text{ times the coefficient of } z_i z_j \text{ in the objective function if } i \neq j. \end{cases}$$

A is a pxr matrix and contains the coefficients of the z_i in the set of constraints.

b is a p-component column vector.

So, the dual according to Zangwill is the following:

Minimize:

$$\underline{u}^t b - \underline{z}^t Q \underline{z} \tag{3.58}$$

Subject to:

$$A^t\underline{u} - 2Qz \geq q \qquad (3.59)$$

$$\underline{u} \geq 0 \qquad (3.60)$$

where $\underline{u}^t$ is a p-component row vector of the dual variables.

The original nonlinear model can be redefined in terms of the general quadratic problem, q, $\underline{z}$, Q, A, and b. Let $\underline{z}^t = (\underline{x}^t, \underline{v}^t, \underline{w}^t, \underline{D}^t)$ be a row vector containing all the decision variables of the original problem with size $[n \times 3(T+1)]x1$, where:

$$\underline{x}^t = (x_1, x_2, \ldots, x_n) \qquad (3.61)$$

$$\underline{v}^t = (v_0, v_1, \ldots, v_T) \qquad (3.62)$$

$$\underline{w}^t = (w_0, w_1, \ldots, w_T) \qquad (3.63)$$

$$\underline{D}^t = (D_0, D_1, \ldots, D_T) \qquad (3.64)$$

Let $q^t = (q_x^t, q_v^t, q_w^t, q_D^t)$ be a row vector containing the coefficients of the linear terms associated with the decision variables x_j, v_t, w_t, and D_t. Its size is $[n \times 3(T+1)]x1$ and the entries are defined as follows:

$$q_x^t = [\hat{a}_1/(1+k)^T, \hat{a}_2/(1+k)^T, \ldots, \hat{a}_n/(1+k)^T] \qquad (3.65)$$

$$q_v^t = [0, 0, \ldots\ldots, 1/(1+k)^T] \qquad (3.66)$$

$$q_w^t = [0,\ 0, \ldots\ldots,\ -1/(1+k)^T] \tag{3.67}$$

$$q_D^t = [1,\ 1/(1+k), \ldots,\ 1/(1+k)^T] \tag{3.68}$$

In order to define Q, it will be helpful to expand the quadratic terms of the objective function. The quadratic differences of the dividends can be expressed as:

$$c_1 \sum_{t=1}^{T} (D_t - D_{t-1})^2 = c_1 D_0^2 - 2c_1 \sum_{t=1}^{T} D_t D_{t-1} + 2c_1 \sum_{t=1}^{T} D_t^2 + c_1 D_T^2 \tag{3.69}$$

The term concerning the difference between terminal wealth and dividends becomes more difficult to expand, but with transformations, it can be shown that:

$$c_2 \{ [\sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T] / c_3 (1+k)^T - \frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^T}] \}^2 =$$

$$c_2 \left[\frac{\sum_{j=1}^{n} \hat{a}_j x_j}{c_3 (1+k)^T} \right]^2 + \frac{2c_2 v_T \sum_{j=1}^{n} \hat{a}_j x_j}{[c_3 (1+k)^T]^2} - \frac{2c_2 w_T \sum_{j=1}^{n} \hat{a}_j x_j}{[c_3 (1+k)^T]^2}$$

$$- \frac{2c_2 [\sum_{j=1}^{n} \hat{a}_j x_j] [\sum_{t=0}^{T} \frac{D_t}{(1+k)^t}]}{c_3 (T+1)(1+k)^T} + c_2 \left[\frac{v_T}{c_3 (1+k)^T} \right]^2 - \frac{2c_2 v_T w_T}{[c_2 (1+k)^T]^2} \tag{3.70}$$

$$- \frac{2c_2 v_T \sum_{t=0}^{T} \frac{D_t}{(1+k)^t}}{c_3 (T+1)(1+k)^T} + c_2 \left[\frac{w_T}{c_3 (1+k)^T} \right]^2 + \frac{2c_2 w_T \sum_{t=0}^{T} \frac{D_t}{(1+k)^t}}{c_3 (T+1)(1+k)^T}$$

$$+ c_2 \left[\frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^t}\right]^2$$

With this in mind, the [n x 3(T+1)] x [n x 3(T+1)] matrix Q may be defined as follows:

$$Q = \begin{bmatrix} Q_{xx} & Q_{vx}^t & Q_{wx}^t & Q_{Dx}^t \\ Q_{vx} & Q_{vv} & Q_{wv}^t & Q_{vv}^t \\ Q_{wx} & Q_{wv} & Q_{ww} & Q_{Dw}^t \\ Q_{Dx} & Q_{Dv} & Q_{Dw} & Q_{DD} \end{bmatrix} \tag{3.71}$$

The entries of (3.71) are also matrices whose elements are the coefficients of the quadratic terms of the decision variables associated with their subindeces. So, knowing those coefficients from (3.69) and (3.70), each one of those matrices is defined as follows:

a) Q_{xx} an (nxn) matrix with entries:

$$(Q_{xx})_{ij} = - \frac{c_2 \hat{a}_i \hat{a}_j}{c_3^2 (1+k)^{2T}} \qquad \begin{array}{l} i = 1,2,\ldots, T \\ j = 1,2,\ldots, T \end{array} \tag{3.72}$$

The minus sign of (3.72) is given by the minus sign associated with the fourth term of the original objective function. In general, all the following entries have the sign changed because of the minus sign of the third and fourth terms of the objective function.

b) Q_{vx} a [(T+1)xn] matrix with non-zero entries just in row T+1

$$(Q_{vx})_{ij} = -\frac{c_2\hat{a}_j}{c_3^2(1+k)^{2T}} \qquad \begin{array}{l} i = T+1 \\ j = 1,2,\ldots, n \end{array} \tag{3.73}$$

$$0 \quad \text{otherwise}$$

c) Q_{wx} is defined exactly as Q_{vx} but with the sign changed:

$$(Q_{wx})_{ij} = \frac{c_2\hat{a}_j}{c_3^2(1+k)^{2T}} \qquad \begin{array}{l} i = T+1 \\ j = 1,2,\ldots, n \end{array} \tag{3.74}$$

$$0 \quad \text{otherwise}$$

d) Q_{Dx} a [(T+1)xn] matrix with entries:

$$(Q_{Dx})_{ij} = \frac{c_2\hat{a}_j}{c_3(T+1)(1+k)^{T+i-1}} \qquad \begin{array}{l} i = 1,2,\ldots, T+1 \\ j = 1,2,\ldots, n \end{array} \tag{3.75}$$

e) Q_{vv} a [(T+1)x(T+1)] matrix with zero entries except one for i = T+1, j = T+1:

$$(Q_{vv})_{ij} = \frac{c_2}{c_3^2(1+k)^{2T}} \qquad \begin{array}{l} i = T+1 \\ j = T+1 \end{array} \tag{3.76}$$

f) Q_{wv} a [(T+1)x(T+1)] matrix with zero entries except one for i = T+1, j = T+1:

$$(Q_{wv})_{ij} = \frac{c_2}{c_3^2(1+k)^{2T}} \qquad \begin{array}{l} i = T+1 \\ j = T+1 \end{array} \tag{3.77}$$

$$0 \quad \text{Otherwise}$$

g) Q_{Dv} a $[(T+1)x(T+1)]$ matrix with non-zero entries just in column T+1:

$$(Q_{Dv})_{ij} = \frac{c_2}{c_3(T+1)(1+k)^{T+i-1}} \qquad \begin{array}{l} i = 1,2,\ldots, T+1 \\ j = T+1 \end{array} \tag{3.78}$$

$$0 \quad \text{Otherwise}$$

h) Q_{ww} is defined exactly as Q_{vv}:

$$(Q_{ww})_{ij} = -\frac{c_2}{c_3^2(1+k)^{2T}} \qquad \begin{array}{l} i = T+1 \\ j = T+1 \end{array} \tag{3.79}$$

i) Q_{Dw} a $[(T+1)x(T+1)]$ matrix defined exactly as Q_{Dv} but with the sign changed:

$$(Q_{Dw})_{ij} = -\frac{c_2}{c_3(T+1)(1+k)^{T+i-1}} \qquad \begin{array}{l} i = 1,2,\ldots, T+1 \\ j = T+1 \end{array} \tag{3.80}$$

$$0 \quad \text{Otherwise}$$

j) Q_{DD} a $[(T+1)x(T+1)]$ symmetric matrix has a more complex form because the terms of (3.69) should also be taken into account. It can be shown that its entries are defined as follows:

$$(Q_{DD})_{ij} = -c_1 - \frac{c_2}{(T+1)^2} \qquad i = 1 \quad j = 1 \tag{3.81}$$

$$= -c_1 - \frac{c_2}{(T+1)(1+k)^{2T}} \qquad i = T+1 \quad j = T+1 \tag{3.82}$$

$$= -2c_1 - \frac{c_2}{(T+1)(1+k)^{i+j-2}} \qquad i = j \quad i \neq 1,\ T+1 \quad j \neq 1,\ T+1 \tag{3.83}$$

$$= -c_1 - \frac{c_2}{(T+1)^2(1+k)^{i+j-2}} \qquad i = 1,2,\ldots,n \quad j = i-1 \tag{3.84}$$

$$= \frac{c_2}{(T+1)^2(1+k)^{i+j-2}} \qquad \text{Otherwise} \tag{3.85}$$

Note that (3.81), (3.82), and (3.83) are entries of the main diagonal, (3.84) of the diagonals next to the main diagonal, and (3.85) are the rest of the entries. The three diagonals in question have special structure because the coefficients associated with the quadratic differences of the dividends are involved in them.

Continuing with the redefinition of our original model as a general quadratic model, the next term is A. Let A be defined as follows:

$$A = \begin{bmatrix} A_{tx} & A_{tv} & A_{tw} & A_{tD} \\ \underline{0} & \underline{0} & \underline{0} & A_D \\ A_x & \underline{0} & \underline{0} & \underline{0} \end{bmatrix} \tag{3.86}$$

The entries of A are also matrices whose elements are the coefficients of the decision variables of the set of constraints. The first row of A corresponds to the coefficients of the budget constraints, the second one to the dividend policy constraints, and the third one to the constraints on the decision variables x_j.

Each one of the entries of A are defined as follows:

a) A_{tx} a[(T+1)xn] matrix whose entries are

$$(A_{tx})_{ij} = -a_{ij} \qquad \begin{array}{l} t = i-1 \\ i = 1,\ldots, T+1 \\ j = 1,\ldots, n \end{array} \tag{3.87}$$

b) A_{tv} a[(T+1)xn] matrix whose entries are all zeros except those of the main diagonal and the diagonal just below it.

$$(A_{tv})_{ij} = 1 \qquad \begin{array}{l} i = 1,2,\ldots, T+1 \\ j = 1,2,\ldots, T+1 \\ i = j \end{array} \tag{3.88}$$

$$= -(1+r_{\ell t}) \qquad \begin{array}{l} i = 2,3,\ldots, T+1 \\ j = 1,2,\ldots, T \\ j = i-1 \\ t = i-1 \end{array} \tag{3.89}$$

c) A_{tw} has a similar structure as A_{tv}. It may be represented as follows:

$$(A_{tw})_{ij} = 1 \qquad i = 1,2,\ldots, T+1 \quad j = 1,2,\ldots, T+1 \quad i = j \tag{3.90}$$

$$= (1+r_{bt}) \qquad i = 2,3,\ldots, T+1 \quad j = 1,2,\ldots, T \quad j = i-1 \quad t = i-1 \tag{3.91}$$

d) A_{tD} is an $[(T+1)x(T+1)]$ identity matrix.

e) A_D is a $[(T+1)x(T+1)]$ diagonal matrix with -1 in all the entries of the diagonal. The minus sign was assigned to the entries because the original constraint (3.52) was an inequality of the type of greater than, and was transformed into the standard form (3.56).

f) A_x is just an (nxn) identity matrix.

All the other entries in A are zeros and they are represented by the zero matrices shown in (3.86).

The right hand side vector b can be defined in terms of the original model as follows:

$$b^t = (b_M^t,\ b_D^t,\ b_x^t) \tag{3.92}$$

Where b_M contains all the budget amounts for the planning period:

$$b_M^t = (M_0, M_1, \ldots M_T) \quad (3.93)$$

Vector b_D is also a (T+1) column vector which contains the minimum dividend payment set by the company's policy. A minus sign has been added because the original inequality (3.52) was transformed into standard form:

$$b_D^t = (-D_{min}, -D_{min}, \ldots, -D_{min}) \quad (3.94)$$

Vector b_x is just a column of ones.

Finally, let $\underline{u}$, the dual variable vector be defined in a way convenient for our case:

$$\underline{u}^t = (\underline{u}_M^t, \underline{u}_D^t, \underline{u}_x^t) \quad (3.95)$$

This $\underline{u}$ is a [n + 2(T+1)] column vector which contains the dual variables corresponding to the budget, dividend policy, and nonnegativity constraints. The variables appear at the left hand side in the original model. $\underline{u}_M$ as well as $\underline{u}_D$ have a size [(T+1) x 1], while $\underline{u}_x$ a size (n x 1).

After redefining every variable of the original model in terms of those used in the general quadratic model, let us rewrite (3.59):

$$A^t\underline{u} - 2Q\underline{z} \geq q \quad (3.59)$$

After substituting the values of A, Q, and q, four inequalities may be obtained which are:

$$A_{tx}^{t}\ \underline{u}_M + A_x^t\ \underline{u}_x - 2[Q_{xx}\ \underline{x} + Q_{vw}^t\ \underline{v} + Q_{wx}^t\ \underline{w} + Q_{Dx}^t\ \underline{D}] \geq q_x \quad (3.96)$$

$$A_{tv}^{t}\ \underline{u}_M - 2\ [Q_{vx}\ \underline{x} + Q_{vv}\ \underline{v} + Q_{ww}^t\ \underline{w} + Q_{Dv}^t\ \underline{D}] \geq q_v \quad (3.97)$$

$$A_{tw}^{t}\ \underline{u}_M - 2\ [Q_{wx}\ \underline{x} + \underline{Q}_{wv}\ \underline{v} + Q_{ww}\ \underline{w} + Q_{Dw}^t\ \underline{D}] \geq q_w \quad (3.98)$$

$$A_{tD}^{t}\ \underline{u}_M + A_D^t\ \underline{u}_D - 2\ [Q_{Dx}\ \underline{x} + Q_{DD}\ \underline{v} + Q_{Dw}\ \underline{w} + Q_{DD}\ \underline{D}] \geq q_D \quad (3.99)$$

These four inequalities are the main objective of our optimality analysis, and each one is analyzed in the following pages. Let us start with (3.96). After substituting for corresponding terms, we obtain n inequalities, whose general form is the following:

$$-\sum_{t=0}^{T} a_{tj}(u_M)_t + (u_x)_j + \frac{2c_2\hat{a}_j}{c_3(1+k)^T}\ [(\sum_{j=1}^{n}\hat{a}_j x_j + v_T - w_T)/c_3(1+k)^T$$

$$(3.100)$$

$$-\frac{1}{T+1}\sum_{t=1}^{T}\frac{D_t}{(1+k)^t}] \geq \frac{\hat{a}_j}{(1+k)^T} \qquad j = 1,2,\ldots, n$$

Where $(u_M)_t$ is the t^{th} term of $\underline{u}_M$, and $(u_x)_j$ the j^{th} term of $\underline{u}_x$. The first term between the brackets of the last element is just the terminal wealth, so (3.100) in shorter notation becomes:

$$-\sum_{t=0}^{T} a_{tj}(u_M)_t + (u_x)_j + \frac{2c_2\hat{a}_j}{c_3(1+k)^T}\ [\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}$$

$$\geq \frac{a_j}{(1+k)^T} \qquad j = 1,2,\ldots, n \qquad (3.101)$$

The primal variable associated with this constraint is the decision variable x_j. If x_j is accepted, (3.100) becomes an equality by the Kuhn-Tucker conditions, and it may be rewritten as:

$$(u_x)_j = \sum_{t=0}^{T} a_{tj}(u_M)_t + \frac{2c_2\hat{a}_j}{c_3(1+k)^T}\left[\frac{c_3}{2c_2} - \frac{G'}{(c_3(1+k)^T} + \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] \qquad j \in S \tag{3.102}$$

Where S is the set of accepted projects.

This equation is further reduced later. In order to do that, (3.97) should be analyzed. Here again, after substituting the corresponding values in it, two general inequalities are obtained:

$$(u_M)_t - (1+r_{\ell t+1})(u_M)_{t+1} \geq 0 \tag{3.103}$$

$$t = 0,1,\ldots, T-1$$

and

$$(u_M)_T + \frac{2c_2}{c_3(1+k)^T}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] \geq \frac{1}{(1+k)^T} \tag{3.104}$$

Where $(u_M)_t$ is the t^{th} element of $\underline{u}_t$.

Inequality (3.103) may be rewritten as:

$$(1+r_{\ell t+1}) \leq \frac{(u_M)_t}{(u_M)_{t+1}} \tag{3.105}$$

This inequality is a logical result and can be compared to the left hand part of (3.26). It shows that the ratio of the dual variables associated with the budget constraints for two subsequent years should be greater than or equal to one plus the lending rate.

Similar relationships are obtained from (3.98):

$$-(u_M)_t + (1+r_{bt+1})(u_M)_{t+1} \geq 0 \qquad t = 0,1,\ldots, T-1 \quad (3.106)$$

and

$$-(u_M)_t - \frac{2c_2}{c_3(1+k)^T}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] \geq -\frac{1}{(1+k)^T} \quad (3.107)$$

Combining (3.105) and (3.106), we obtain the classical relationship among the lending and borrowing rates, and the budget constraint dual variables:

$$(1+r_{\ell t+1}) \leq \frac{(u_M)_t}{(u_M)_{t+1}} \leq (1+r_{bt+1}) \quad (3.108)$$

On the other hand, it may be seen that both inequalities (3.104) and (3.107) combined become one equation which may be written as follows:

$$(u_M)_t = \frac{2c_3}{c_3(1+k)^T}\left[\frac{c_3}{2c_2} - \frac{G'}{c_3(1+k)^T} + \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right. \quad (3.109)$$

We have found a relationship for $(u_M)_T$ as a function of the parameter c_2, c_3, the discount rate, the terminal wealth and the stream of dividends. In the basic horizon model of Weingartner, $(u_M)_T$ is equal to one.

Equation (3.109) is now substituted into (3.102) and we obtain:

$$(u_x)_j = \sum_{t=0}^{T} a_{tj}(u_M)_t + \hat{a}_j(u_M)_T \qquad j \in S \tag{3.110}$$

Thie equation is similar to (3.35) of the linear model.

From complementary slackness (1), we know that if project j is <u>partially</u> accepted, $(u_x)_j$ becomes zero:

$$\sum_{t=0}^{T} a_{tj}(u_M)_t + \hat{a}_j(u_M)_t = 0 \qquad j \in S'' \tag{3.111}$$

where S´´ is the set of projects partially accepted. What (3.111) means is that if the right hand side of (3.53) were allowed to be one unit more, the net increment of the acceptance of project j would be zero. Put it in another way, even though project j could be undertaken several times, the model would still select it fractionally.

From (3.99), we may obtain:

$$(u_M)_0 - (u_D)_0 - \frac{2c_2}{T+1}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] + 2c_1(D_0+D_1) \geq 1 \tag{3.112}$$

$$(u_M)_t - (u_D)_t - \frac{2c_2}{(T+1)(1+k)^t}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right]$$

$$+ 2c_1(D_{t-1} + 2D_t + D_{t+1}) \geq \frac{1}{(1+k)^t} \tag{3.113}$$

$$t = 1, 2, \ldots, T-1$$

$$(u_M)_t - (u_D)_T - \frac{2c_2}{(T+1)(1+k)^T}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right]$$

$$+ 2c_1\ [D_{T-1} + D_T] \geq \frac{1}{(1+k)^T} \qquad t = T \tag{3.114}$$

Knowing from (3.109) that:

$$-\frac{2c_2}{(T+1)(1+k)^T}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] = \frac{c_3}{T+1}\left[(u_M)_T - \frac{1}{(1+k)^T}\right] \tag{3.115}$$

We may rewrite (3.114) in a simpler way:

$$(u_M)_T\ \left[1 + \frac{c_3}{T+1}\right] - (u_D)_T + 2c_1(D_{T-1}+D_T) \geq \frac{1}{(1+k)^T}\left[1 + \frac{c_3}{T+1}\right] \tag{3.116}$$

Note that if D_t is strictly greater than D_{min}, its associated slack will be nonzero and from complementary slackness, $(u_D)_t$ is zero. In addition, given that D_t is greater than D_{min}, its associated dual constraint will be tight. If this is the case, (3.112), (3.113) and (3.116) may be rewritten as follows:

$$(u_M)_0 - \frac{2c_2}{T+1}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] + 2c_1(D_0+D_1) = 1$$

$$t = 0 \qquad (3.117)$$

$$D_0 > D_{min}$$

$$(u_M)_t - \frac{2c_2}{(T+1)(1+k)^t}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] +$$

$$+ 2c_1(_{t-1} + 2D_t + D_{t+1}) = \frac{1}{(1+k)^t} \qquad (3.118)$$

$$t = 1,2,\ldots, T-1$$

$$D_t > D_{min}$$

$$(u_M)_T\left[1 + \frac{c_3}{T+1}\right] + 2c_1(D_{T-1} + D_T) = \frac{1}{(1+k)^T}\left[1 + \frac{c_3}{T+1}\right] \qquad (3.119)$$

$$t = T$$

$$D_T > D_{min}$$

Note that the inverse of the "horizon pay-out" is involved in this equation.

The objective function (3.56) may be expressed in terms of the variables of our nonlinear model as follows:

Minimize:

$$\sum_{t=0}^{T} (u_M)_t M_t - \sum_{t=0}^{T} (u_D)_t D_{min} + \sum_{j=1}^{n} (u_x)_j - \underline{z}^t Q \underline{z} \qquad (3.120)$$

The $\underline{z}^t Q \underline{z}$ are just the quadratic terms of the primal objective function.

Using (3.110) we obtain:

$$\sum_{j=1}^{n} (u_x)_j = \sum_{j \varepsilon S} \left[\sum_{t=0}^{T} a_{tj} (u_M)_t + \hat{a}_j (u_M)_T \right] \qquad (3.121)$$

$$j \ \varepsilon \ S$$

and substituting it in the objective function:

Minimize:

$$\sum_{t=0}^{T} (u_M)_t [M_t + \sum_{j \varepsilon S} a_{tj}] + \sum_{j \varepsilon S} \hat{a}_j (u_M)_T$$

$$- \sum_{t=0}^{T} (u_D)_t D_{min} - \underline{z}^t Q \underline{z} \qquad j \ \varepsilon \ S \qquad (3.122)$$

This equation may be compared with (3.44) of the linear model. Recall that (3.44) is independent of parameters c_1 and c_2. However, for the nonlinear case, (3.122) depends on all three parameters, c_1, c_2, and c_3, which are embedded in the Q matrix.

Range of Parameters of the Nonlinear Model

Given the way in which the parameters are defined in the nonlinear model, their interpretation is slightly different from that of the linear case. Parameter c_1, for example, is the weight assigned to the quadratic differences of dividends, while in the first case to the slope of a linear regression on dividends. Parameter c_2 in the first case worked against the variability between terminal wealth and dividends, while in the present model against the quadratic difference of the same term. Parameter c_3 is basically defined in the same way. Because of these reasons, the range of the parameters of the linear model may also be different from those of the nonlinear case, except perhaps for parameter c_3.

The range for parameter c_1 can be obtained from an analysis of either (3.117), (3.118), or (3.119). This simple analysis is based on the assumption that the dividend payments are greater than D_{min}. In such a case, the slack variables associated with the dividend policy constraints (3.52) would be nonzero, and therefore their corresponding duals $(u_D)_t$, zero. Moreover $(u_M)_t$ is always nonnegative and because of this, (3.117), (3.118), and (3.119) can be rewritten as follows:

$$c_1 \leq \frac{1 + \frac{2c_2}{T+1}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right]}{2(D_0 + D_1)} \qquad (3.123)$$

$$D_0 > D_{min}$$

$$c_1 \leq \frac{\frac{1}{(1+k)^t} + \frac{2c_2}{(T+1)(1+k)^t}\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right]}{2(D_{t-1} + 2D_t + D_{t+1})} \qquad (3.124)$$

$$t = 1,2,\ldots, T-1$$

$$D_t > D_{min}$$

$$c_1 \leq \frac{\frac{1}{(1+k)^T}\left[1 + \frac{c_3}{T+1}\right]}{2(D_{t-1} + D_t} \qquad (3.125)$$

$$D_T > D_{min}$$

The search for finding the range of c_1 may start with (3.125) because it is the less complex inequality of those shown above. Setting in advance a value for the "horizon pay-out," reasonable values for D_{T-1} and D_T can be assumed. These assumptions should be based on the decision-maker's subjective judgment about the company's potential earnings within the planning period as well as about the stockholders' income desires. With those assumptions on hand, obtaining an upper bound for c_1 represents no problem. The lower bound for c_1 basically is zero, an extreme case, in which the model lets the dividends vary freely.

Useful information may also be obtained from (3.123) and (3.124). Given that the objective of the company according to our model is to have the difference between terminal wealth and dividends as low as possible, this difference might be assumed zero:

$$c_1 \leq \frac{1}{2(D_0 + D_1)} \qquad D_0 > D_{min} \tag{3.126}$$

$$c_2 \leq \frac{1}{2(D_{t-1} + 2D_t + D_{t-1})(1+k)^t} \tag{3.127}$$

$$D_t > D_{min}$$

$$t = 1,2,\ldots, T-1$$

Here again, assumptions should be made about the dividends in question. Even though helpful, the use of (3.123) and (3.124) is not as strongly suggested by the author as (3.125) because this inequality requires fewer assumptions than the first two.

The range for parameter c_2 can be obtained from analysis of (3.109). Given that $(u_M)_T$ should always be nonnegative, we can say that:

$$\frac{c_3}{c_2} \geq 2\left[\frac{G'}{c_3(1+k)^T} - \frac{1}{T+1}\sum_{t=0}^{T}\frac{D_t}{(1+k)^t}\right] \tag{3.128}$$

An initial idea of the value of the term between brackets is needed. Ideally, it would be zero, but conjecturing in advance, it will seldom be obtained in this nonlinear model. The difference value depends upon the investments and dividends under analysis.

The range of c_3 is set using the same criteria as in the linear model: the ratio $(T+1)/c_3$ was defined before as some "horizon pay-out" ratio given that at the optimum it is equal to the ratio of the time-zero present worth of dividend payments to the time-zero present worth of terminal wealth. As stated before, this ratio may be set up as an objective and so the value or values of c_3.

Summary

In this chapter, two models, one linear and one non-linear, have been presented. Through the use of duality theory, an analysis has been made of their expected results at optimality conditions. In the next chapter, both models are tested under similar conditions, and the conclusions obtained in this chapter will be useful in evaluating the performance of the models.

CHAPTER IV

COMPUTATIONAL RESULTS

Problem Selection

In order to study the behavior of our models, two sets of projects are used. Both sets of projects follow nearly the same pattern of cash flows, but they differ in the rates of return of the projects. The rates of return of most of the projects of the first set are near 10 percent and the discount rate used is 9.5 percent. This peculiar set of rates of return was selected after finding that the linear model has a low sensitivity regarding the projects' rate of return. This will be seen later in the second set, in which the rates of return of the projects fluctuate from 7 percent to 13 percent; when a discount rate of 10 percent is used, results which are completely independent of c_1 and c_2 are obtained. The nonlinear model does not exhibit such behavior, and for both sets of projects, different results are obtained for each parameter value.

In the first set of projects, 9 percent and 10.1 percent lending and borrowing rates, respectively, are used for the same reason explained above, while in the second case, 8 percent and 12 percent are used. A horizon of 5 years is taken for both cases. The two sets of projects are shown in Figure 4.1 and 4.2.

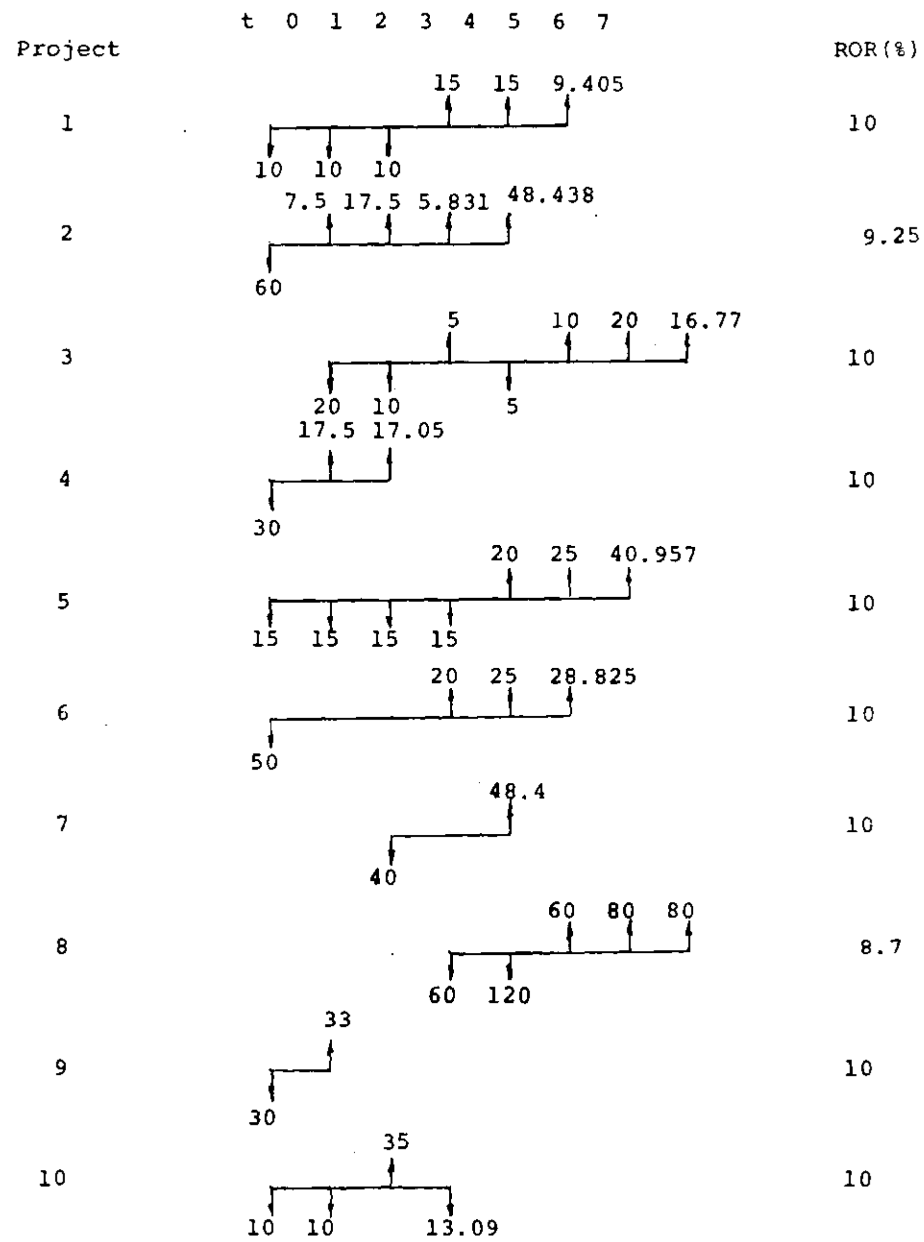

Figure 4.1. Cash Flows of Project Set 1.
NOTE: Budget constraint for t = 0 is 70,000 and for other years 15,000. All the cash flows are in thousands of dollars.

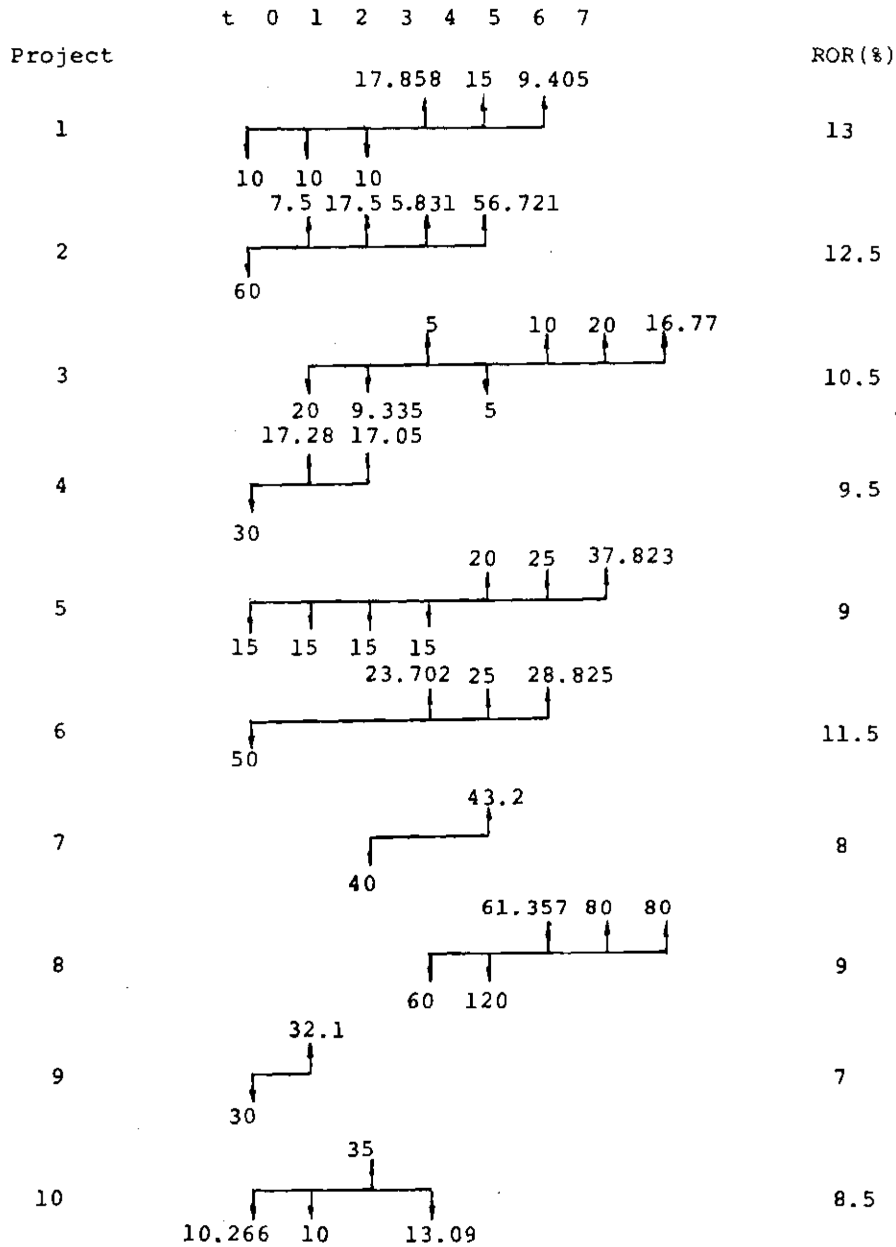

Figure 4.2. Cash Flows of Project Set 2.
NOTE: Budget constraint for t = 0 is 70,000, and for other years 15,000. All the Cash flows are in thousands of dollars.

The way in which the analysis of the behavior of the models is presented in this chapter is as follows: using four predetermined values of the parameters c_1, c_2, and c_3, 64 computer runs are made (a 4^3 factorial experiment) for each set of projects, for each model. Some runs for the linear model are not made because they are infeasible according to the rules obtained in the "range of parameters" of the last chapter. With the help of plots of several responses of the models against c_1, c_2, and c_3, a complete analysis is made and conclusions are drawn. Finally, the optimality conditions in Chapter III are checked to investigate whether or not they agree with the numerical results. Before getting into the analysis mentioned above, an explanation of the algorithms used is presented.

Description of Algorithms

Linear Model

The linear programming code EZLP (11) was used to make the runs of the linear model because of the advantages in data input which that program has. However, given that 40 runs should be made using different trios of parameters values which produce different coefficients in the objective function as well as in the constraints, a FORTRAN program called MODEL 1 was written for computing the various coefficients and for creating the data input file for EZLP. Figure 4.3 shows the main steps of the software used in the linear model runs.

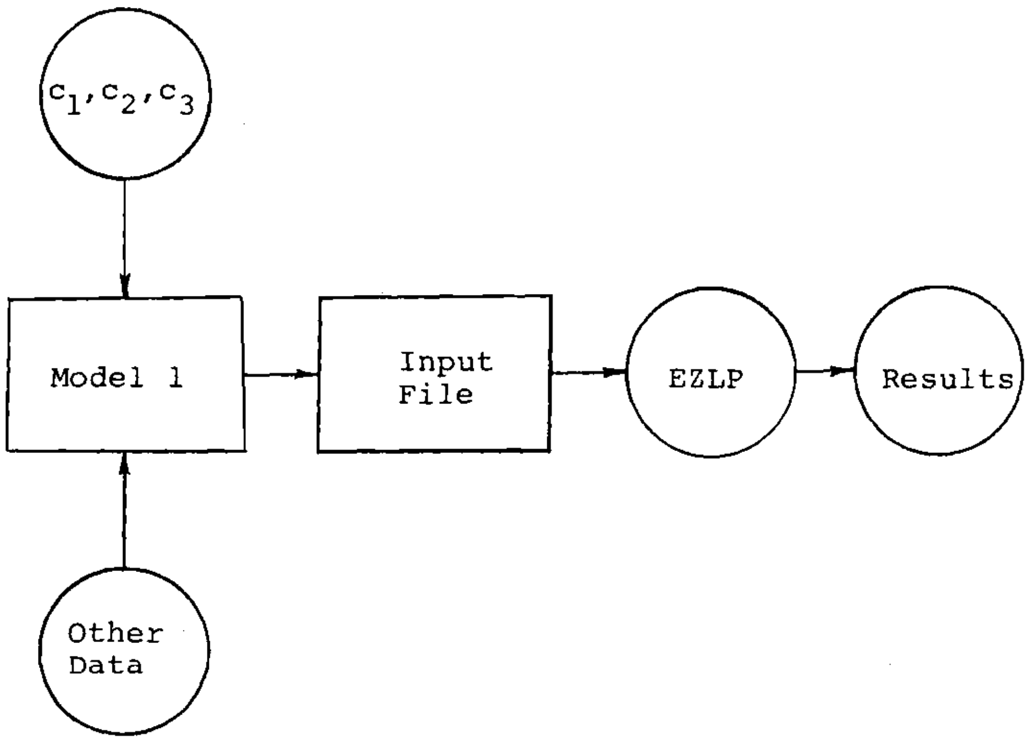

Figure 4.3. Main Steps of the Software of the Linear Model.

The steps shown in Figure 4.3 represent about 20 commands which should be typed in the interactive terminal for obtaining one set of results. To avoid doing that each time a run is made, a SUBMIT file containing all the commands in question is used. Its code is shown in Appendix A.

MODEL 1 has several characteristics which should be explained further:

a) MODEL 1 requires the use of a data file containing the project cash flows, the budget constrains, and the borrowing and lending rates. Only projects with cash flows between t=0 and t=10 are allowed. Thus the cash flow information is contained in a matrix of size (11 x 11).

b) The parameters c_1, c_2, c_3, the length of the horizon, the minimum dividend payment, and the discount rate used are input within the program using the command DATA. The maximum length of the horizon is 10 years.

c) MODEL 1 adds the different coefficients related to the same variable and gives the sum, simplifying the objective function.

d) As should have been anticipated, c_3 is not allowed to be zero.

The complete code of MODEL 1 is presented in Appendix B.

<u>Nonlinear Model</u>

Among the nonlinear packages available at Georgia Tech, the best program which satisfies the requirements of the nonlinear model is Beale's Quadratic Programming Method within the MPOS package (16). Beale's method solves quadratic problems of the general form shown in Chapter III. It works well, according to the MPOS manual, with quadratic problems containing a high number of zero entries in the matrix Q (see 3.55). Further analysis of the advantages of Beale's method are beyond the scope of this thesis.

As in the linear case, 64 runs are made at four different levels for each one of the parameters. Here again, a FORTRAN program called MODEL 2, is set up for computing the model coefficients.

MPOS has several options for data input. Given the number of variables involved in our case, as well as the high number of zeros entries in matrix Q, the PACKAGE form (16) was used in which just the non-zero coefficients are input (all others are assumed zero). MODEL 2 is set up so that it gives as output a data file which can be used as input, according to the PACKAGE form. Basically, this data input form has the following characteristics:

a) There are as many lines as non-zero coefficients in the quadratic problem.

b) Each line contains the row number (row zero means the objective function) to which the coefficient belongs, the variable number related to that coefficient, and the coefficient itself. A double variable means that the coefficient in question belongs to a quadratic term.

As in the linear case, a SUBMIT file is used to make the runs in order to avoid the typing of nearly 30 commands in every run. The main steps of the software used are shown in Figure 4.4, and the SUBMIT file is listed in Appendix C. MODEL 2 had exactly the same data requirements as well as limitations as MODEL 1. A copy of its FORTRAN code is in Appendix D.

Linear Model Results

Range of Parameters

The relationships established in Chapter III for determining the ranges of the parameters c_1, c_2, and c_3 were observed. However, that was not sufficient and several additional runs were made for finding the best region of interest.

For the project set shown before, and using (3.33), (3.47), and (3.48), it is easy to show that:

1) c_1 should be less than or equal to 17.5
2) c_2 should be less than or equal to 6, and also less than c_3.

The author found that for c_1, a region considered of interest is between 0.5 and 10, while for c_2, between 0.5 and 4. Thus, the four levels selected for c_1 and c_2 turn out to be 0.5, 2, 8, 10, and 0.5, 1.5, 2.5, 4, respectively.

The range of c_3 is found using the criteria discussed in the last chapter. The ratio $(T+1)/c_3$ was defined as the "horizon pay-out" and with this in mind, the value of c_3

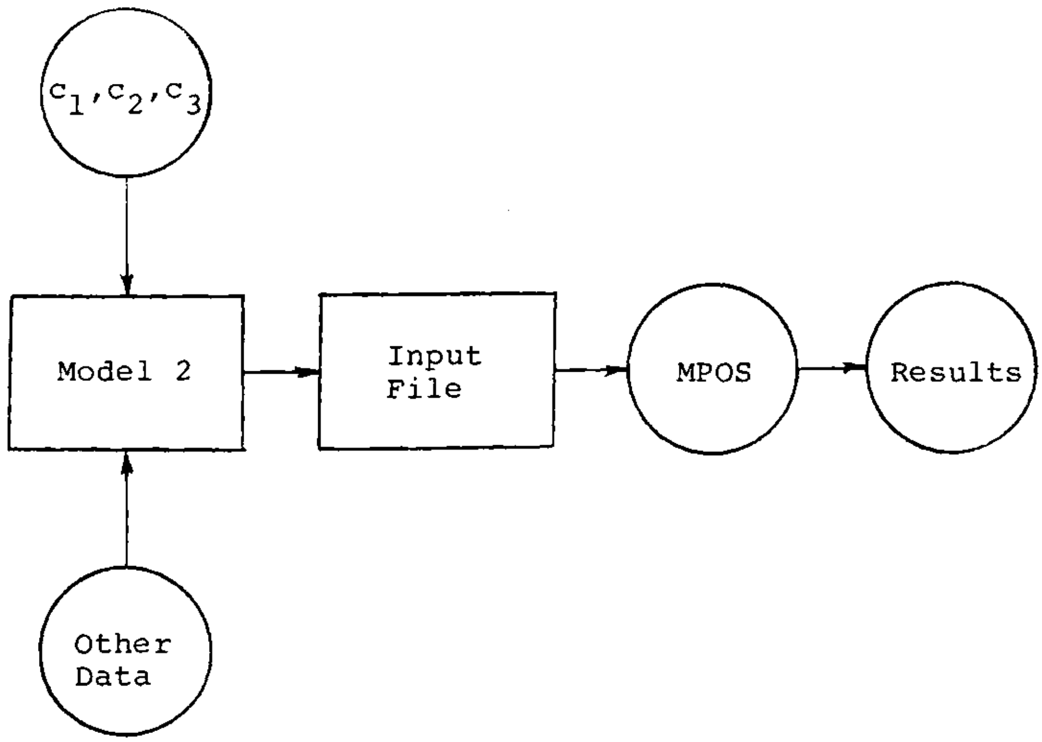

Figure 4.4. Main Steps of the Software of the Nonlinear Model.

should be chosen in such a way that it satisfies the company's policy requirements. The author tried ratios from 1.2 to 6 with excellent results. The ratio values imply values of c_3 from 1 to 5. Therefore, the four values of c_3 used in the runs are 1, 2, 4, and 5.

Performance Criteria

It is clear that in order to evaluate the performance of the model, some criteria should be defined. These criteria represent the information which the decision-maker should analyze to determine whether or not the results obtained, given certain values of c_1, c_2, c_3, satisfy the company's policy as well as the stockholders' desires. Another reason why we must define some criteria is the fact that there are many different results, depending on different sets of values for c_1, c_2, and c_3. Every one of those results may be "best" depending upon the criteria as well as the objectives of the model user.

Three general criteria have been selected by the author, who considers that they give sufficient information for evaluating the results for research purposes. These criteria are the following:

a) The terminal wealth G', which contains information about the company's gross earnings coming from the projects selected as well as from the lending activities within the horizon.

b) The average of dividend payments, defined as:

$$\bar{D} = \sum_{t=0}^{T} \frac{D_t}{T+1} \tag{4.1}$$

This contains information about the average level of the payments. This is not enough for a stockholder, because sometimes he might prefer a low average level of dividends with little variability, to a high average level with greater fluctuations. So we also define:

c) The standard deviation of the stream of dividends within the horizon, defined as follows:

$$s = \frac{\sum_{t=0}^{T} D_t^2 - (T+1)\bar{D}^2}{T} \tag{4.2}$$

Basically, the analysis presented in the following pages is based on these three criteria. It consists of a study of the changes of those three model responses given different values of c_1, c_2, and c_3. From this parametric analysis, a complete set of different solutions is available for the decision-maker.

Parametric Analysis

Using the levels selected before for parameters c_1, c_2, and c_3, 40 runs were made for each set of projects. A group of selected results are shown in Appendices E and F. Also, the levels of the terminal wealth, average dividend, and standard deviation of dividend are shown in Tables 4.1, 4.2, and 4.3 for the first project set, and in Tables 4.4, 4.5, and 4.6 for the second set.

Table 4.1. Terminal Wealth for Project Set 1, Linear Model.

c_1	c_2	c_3 = 1	2	4	5
0.5	0.5	28294	50687	81271	92355
	1.5	-	50687	81271	92425
	2.5	-	-	81232	92396
	4	-	-	-	92384
2	0.5	28294	50687	81267	92425
	1.5	-	50687	81267	92425
	2.5	-	-	81267	92425
	4	-	-	-	92396
8	0.5	28294	50682	81254	92405
	1.5	-	50664	81246	92395
	2.5	-	-	81246	92395
	4	-	-	-	92404
10	0.5	28294	50687	81264	92425
	1.5	-	50679	81254	92404
	2.5	-	-	81254	92404
	4	-	-	-	92404

Table 4.2. Standard Deviation for Project Set 1, Linear Model.

c_1	c_2	c_3 = 1	2	4	5
0.5	0.5	26281	22710	17222	14634
0.5	1.5	-	22707	17222	14631
0.5	2.5	-	-	13950	12599
0.5	4	-	-	-	13603
2	0.5	26281	22707	17221	14631
2	1.5	-	22707	17221	14630
2	2.5	-	-	17221	14633
2	4	-	-	-	12599
8	0.5	31798	27400	20149	17501
8	1.5	-	17885	13950	14599
8	2.5	-	-	13950	12599
8	4	-	-	-	13592
10	0.5	26080	21899	16214	14960
10	1.5	-	27399	20149	15097
10	2.5	-	-	20149	15097
10	4	-	-	-	16978

Table 4.3. Average Dividend for Project Set 1, Linear Model.

c_1	c_2	c_3 = 1	2	4	5
0.5	0.5	22947	20119	16116	14657
	1.5	-	20118	16116	14657
	2.5	-	-	16075	14622
	4	-	-	-	14625
2	0.5	22947	20118	16116	14657
	1.5	-	20118	16116	14657
	2.5	-	-	16116	14657
	4	-	-	-	14623
8	0.5	22947	20086	16095	14637
	1.5	-	20051	16075	14621
	2.5	-	-	16075	14621
	4	-	-	-	14623
10	0.5	22947	20133	16113	14657
	1.5	-	20086	16095	14629
	2.5	-	-	16095	14629
	4	-	-	-	16439

Table 4.4. Terminal Wealth for Project Set 2, Linear Model.

c_1	c_2	c_3 = 1	2	4	5
0.5	0.5	30290	53120	85130	96780
	1.5	-	53120	85130	96770
	2.5	-	-	85130	96770
	4	-	-	-	96770
2	0.5	30290	53120	85190	96780
	1.5	-	53120	85130	96770
	2.5	-	-	85130	96770
	4	-	-	-	96770
8	0.5	30290	53120	85130	96770
	1.5	-	53120	85130	96770
	2.5	-	-	85130	96770
	4	-	-	-	96770
10	0.5	30290	53120	85190	96780
	1.5	-	53120	85190	96780
	2.5	-	-	85130	96770
	4	-	-	-	96770

Table 4.5. Standard Deviation for Project Set 2, Linear Model.

c_1	c_2	c_3: 1	2	4	5
0.5	0.5	28458	23877	17464	15137
	1.5	-	23877	17461	15134
	2.5	-	-	17461	15134
	4	-	-	-	15134
2	0.5	28458	23877	17464	15137
	1.5	-	23877	17461	15134
	2.5	-	-	17461	15134
	4	-	-	-	15134
8	0.5	28458	23877	17461	15134
	1.5	-	23877	17461	15134
	2.5	-	-	17461	15134
	4	-	-	17461	15134
10	0.5	28458	23877	17464	15137
	1.5	-	23877	17464	15137
	2.5	-	-	17461	15137
	4	-	-	-	15137

Table 4.6. Average Dividend for Project Set 2, Linear Model.

c_1	c_2	c_3 = 1	2	4	5
0.5	0.5	23801	28838	16678	15165
	1.5	–	28838	16677	15163
	2.5	–	–	16677	15163
	4	–	–	–	15163
2	0.5	23801	28838	16678	15165
	1.5	–	28838	16677	15163
	2.5	–	–	16677	15163
	4	–	–	–	15163
8	0.5	23801	28838	16677	15163
	1.5	–	28838	16677	15163
	2.5	–	–	16677	15163
	4	–	–	–	15163
10	0.5	23801	28838	16678	15165
	1.5	–	28838	16678	15165
	2.5	–	–	16677	15163
	4	–	–	–	15163

It should be noted that in the tables mentioned above there are 24 entries left blank. Those entries belong to sets of parameters in which c_2 is not less than c_3, violating (3.33). Runs for $c_2 = 4$ and $c_3 = 4$, are also suppressed because the solution is unaffected by them and so there is no interest in showing them.

First, let us consider project set 1. Looking at Tables 4.1, 4.2, and 4.3, it may be seen that c_1 as well as c_2 do not cause any considerable effect on G', s or $\bar{D}$, and that the parameter which controls the response is c_3. Even though c_1 and c_2 do not affect those three responses, they obtain different stream of dividend payments for the same levels of G', s, and $\bar{D}$, which is an interesting result (see Appendix E). The decision-maker thus has several alternatives among which he may chose.

The reason why neither c_1 nor c_2 affect G', s, and $\bar{D}$ is clearly seen in (3.44). We may observe that the dual objective function, written in this way, is independent of c_1 and c_2.

Therefore, the main subject of our analysis is c_3. Plots of G', s, and $\bar{D}$ against c_3 are shown in Figure G-1, G-2, and G-3 in the appendix. Figure G-1 shows G' vs. c_3. It may be noted that the terminal wealth increases when c_3 increases. Note that saying that c_3 increases is equivalent to saying that the "horizon pay-out" is lowered, which implies the logical result of lower dividends paid and higher "retained earnings" (terminal wealth).

Opposite effects of c_3 on s and $\bar{D}$ are found in Figures G-2 and G-3. Both the standard deviation and the average dividend decrease as c_3 is increased. The last is exactly what has been expected. The reader should remember that c_3 is a parameter which helps to minimize the difference between terminal wealth and dividend payments. Another viewpoint can be obtained by rewriting (3.28) in the following way:

$$-(1+k)^T c_3 f_T = i + c_2 - c_3 \tag{4.3}$$

This term appears in the dual objective function shown in (3.44), and after its substitution, the dual objective function becomes:

$$\sum_{t=0}^{T} [M_t + \sum_{j\varepsilon S'} \hat{a}_j - D_{min}] f_t + \sum_{j\varepsilon S'} \hat{a}_j f_T + KD_{min} [\frac{i+c_2-c_3}{T+1}] \tag{4.4}$$

$$+ KD_{min}$$

Given the structure of the dual objective function stated as above, and knowing from (3.24) that i is constrained to be nonpositive, we may say that the optimal condition for i is to be as negative as possible, subject to (4.3). One the other hand, there is nothing in (4.3) which can force i to be zero, so it may be concluded that i should always be a nonpositive number different from zero. In these circumstances, the primal constraint (3.5) can be written as an equality:

$$\frac{G'}{(1+k)^T c_3} = \frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^t} \tag{4.5}$$

It is clear from (4.5) that when c_3 is increased, either G′ should increase too, or the dividend payments should decrease. The author found that the latter is true in every run for both projects.

Optimality Analysis

From a cursory look at the Appendix E, we may note that D_0 as well as D_5 are always equal to D_{min}. This is well understood if we examine (3.39), which is rewritten here:

$$h_t = \frac{1}{(1+k)^t} [1- \frac{c_3(1+k)^T}{T+1} f_T + \frac{c_3}{T+1}] - f_t + [\frac{T}{2} - t][\frac{c_1+g}{C}] \tag{4.6}$$

It should be noted that if h_t is strictly less than zero, by the Kuhn-Tucker conditions D_t will be equal to D_{min}. This is exactly what happens when t = 0 and t = T in (4.6). When T = 0, f_0, which we know is greater than any other f_t, dominates (4.6) and makes h_0 negative. When t = T, f_T is helped by the last term $(-\frac{T}{2})(c_1+g)/C$, which is also negative as we may see in a numerical example. Let's take at random any of the runs of the project set 1: from Table E-2 in the Appendix E-2, we rewrite:

$c_1 = 2 \quad c_2 = 0.5 \quad c_3 = 2$

$x_3 = 1 \quad x_4 = 1 \quad x_5 = 0.7267 \quad x_9 = 0.4701 \quad x_{10} = 1$

$D_0=5{,}000 \quad D_1=7{,}109 \quad D_2=46{,}140 \quad D_3=5{,}246 \quad D_4=5{,}000 \quad D_5=5{,}000$

$w_3 = 56{,}450 \quad w_4 = 42{,}610 \quad w_5 = 8{,}745$

Zero otherwise

The corresponding dual variables for this run (not shown in the appendix) turn out to be:

$f_0 = 1.015 \quad f_1 = 0.9227 \quad f_2 = 0.8391 \quad f_3 = 0.7622$

$f_4 = 0.6923 \quad f_5 = 0.6288$

$h_0 = -0.001816 \quad h_4 = -0.0005994 \quad h_5 = 0.001947$

$\ell_3 = 71.81 \quad \ell_4 = 4.686 \quad \ell_{10} = 14.37$

$g = -1.929$

$i = -\ 0.4798$

Zero otherwise

In addition we know that the borrowing and lending rates are 9 percent and 10.1 percent, respectively, the discount rate 9.5 percent, and the horizon 5 years. From (3.25), we know that C = 17.5. Using (4.6) for t=0 and t=5, we obtain:

$$h_0 = [1 - \frac{(2)(1.095)^5}{6}(0.6288) + \frac{2}{6}] - 1.015 + \frac{5}{2}[\frac{2-1.929}{17.5}]$$

$$= -0.001484 \approx -0.001816$$

$$h_5 = \frac{1}{1.095^5}\ [1 - \frac{(2)(1.095)^5}{6}(0.6288) + \frac{2}{6}] - 0.6288 - \frac{5}{2}[\frac{2-1.929}{17.5}]$$

$$= -0.001652 \approx -0.001947$$

The approximations are due to the loss of accuracy of the EZLP code in solving the problem.

The rest of the optimality conditions given in Chapter III may also be checked. Let's consider (3.26), the ratio of the dual variables related to the primal budget constraints:

$$1.09 \leq \frac{f_t}{f_{t+1}} \leq 1.101 \tag{4.7}$$

It is easy to show that:

$$\frac{f_0}{f_1} = 1.100 \quad \frac{f_1}{f_2} = 1.099 \quad \frac{f_2}{f_3} = 1.100$$

$$\frac{f_3}{f_4} = 1.100 \quad \frac{f_4}{f_5} = 1.100$$

Note that all of these satisfy (4.7). Furthermore, all of them are nearly at the upper bound, which implies borrowing activities, as may be seen in the results.

We may also check (3.29):

$$i = (2)[1-(1.095)^5(0.6288)]-0.5 = -0.47976 \simeq -0.4798$$

And also (3.35) for all the projects which are selected:

$$\ell_3 = (32{,}251.2)(0.6288)-(20{,}000)(0.9227)-(10{,}000)(0.8391)$$
$$+ (5000)(0.7622)-(5000)(0.6923)+(10{,}000)(0.6288)=72.05$$
$$\simeq 71.81$$

$$\ell_4 = -(30{,}000)(1.015)+(17{,}500)(0.9227)+(17{,}050)(0.8391)=3.906$$
$$\simeq 4.686$$

$$\ell_{10} = -(10{,}000)(1.015)-(10{,}000)(0.9227)+(35{,}000)(0.8391)$$
$$-(13{,}090)(0.7622) = 14.32 \approx 14.37$$

Regarding Figure G-1, G-2, and G-3, the tradeoff between terminal wealth and the average dividend is clearly seen. The decision-maker may choose the point at which both satisfy his interest. The difficulty is that the standard deviation tends to follow $\bar{D}$. The standard deviation values are as high as $\bar{D}$ for all values of c_3. This constitutes one of the drawbacks of the linear model.

Regarding the projects, we may note in Appendix E that basically the same projects are selected in each one of the runs. The projects which are selected are 3, 4, 5, 9, and 10. Project 3 is fully accepted in all the runs, while the others are partially selected in one run and fully in others. Even though there are exceptions, we may say that when c_1 increases, there is a tendency to accept projects fully. However, there is no clear behavior of the other two parameters regarding the acceptance of projects. There is a clear trend for borrowing at low values of c_3 and for lending at high. For example, in Table E-1, there are borrowing activities at 5 = 3,4 and 5, while in E-4 just in t = 3,4 and money is lent at t=5.

As commented before, another set of projects, set 2 was analyzed using the linear model. This was done because the author found that the linear model lacks sensitivity regarding the project's rate of return. As in the first set of projects, c_1 and c_2 do not have any significant effect on G', s and $\bar{D}$ (see Tables 4.4, 4.5, and 4.6). The difference is that while in the first project set, varying c_1 and c_2 results in different streams of dividends (without affecting s and $\bar{D}$); in the second no such effect occurs.

This may be easily explained. Given a series of projects, whose rates of return rank from 7 percent to 13 percent, there are just two projects selected (one and two) regardless of the value of the parameters. These projects are those with the high rates of return (13 percent and 12.5 percent, respectively). That means that once the model has found those two projects, it does not consider any other alternatives. This becomes another drawback of the linear model.

Nonlinear Model Results

Ranges of Parameters

As commented earlier, the relationships for the parameter range determination in Chapter III provide a good starting point. However, several runs were made in order to determine exactly the region of interest. In Chapter III, it is explained that the "horizon pay-out" should be established in advance by decision-maker. The author tried ratios from 0.3 to 0.6 with excellent results. These ratios imply values for c_3 from 10 to 20, given that the horizon length is 5 years. So, four levels were selected for c_3: 10, 13, 16, and 20.

In order to find the range of c_1, inequality (3.125) was used. Using an intermediate value of the range established in advance for the "horizon pay-out" (0.45) and assuming D_T and D_{T-1} equal to D_{min}, it is easy to show that c_1 should be less than or equal to 8.45×10^{-5}. For practical purposes, the upper bound of c_1 was set equal to 1.0×10^{-4}. After several runs, the author found that the lower bound should be around 1.0×10^{-7}. Values of c_1 less this seemed to have no effect on the results. So, four levels for c_1 were selected to cover the range of c_1: 1.0×10^{-7}, 1.0×10^{-6}, 1.0×10^{-5}, and 1.0×10^{-4}.

Once c_1 and c_3 were fixed, we may use (3.128) for determing c_2. A predetermined value for the difference between terminal wealth and dividends should be given. As commented in Chapter III, the ideal value for this difference is zero, which however is seldom obtained. A good assumption is to estimate the difference between 1,000 and 5,000. So, using the maximum value of c_3 for 1,000, and the minimum for 5,000, we obtain from (3.128):

$$c_2 \leq \frac{20}{2(1000)} = 0.01$$

$$c_2 \leq \frac{10}{2(5000)} = 0.001$$

With this in mind, four levels of c_2 were selected within that range: 0.001, 0.003, 0.006, and 0.01.

Performance Criteria

The same measurements used in the evaluation of the results of the linear model are applied for the nonlinear case. The three criteria, G', s, and $\bar{D}$ are defined in the same way, and they are the core of our analysis.

Parametric Analysis

Exactly as in the linear model, with four levels of c_1, c_2, and c_3, 64 runs are made to investigate the behavior of the nonlinear model. Here also, both sets of projects, 1 and 2, are used to see if the rates of return of the projects have much effect on the model response, as happened in the linear case.

A group of selected results are shown in Appendices H and I for both project sets. In addition, the corresponding levels of terminal wealth, average dividend, and standard deviation of dividend of project set 1 are summarized in Tables 4.7 through 4.9, and in Tables 4.10 through 4.12 for project set 2.

All sets of parameters result in feasible values, contrasting with the linear model. Let's consider first the project set 1. From a brief analysis of Tables 4.7, 4.8, and 4.9, we may note that the three parameters have some effects.

Parameter c_1 is the only one whose effects on G' and $\bar{D}$ are not considerable. This is clearly seen in Figures J-1 and

Table 4.7. Terminal Wealth for Project Set 1, Nonlinear Model.

c_1	c_2	c_3: 10	13	16	20
10^{-7}	.001	142667	142667	147324	152449
	.003	128097	139331	156651	162199
	.006	96672	142426	149995	157455
	.01	63804	100363	128500	157929
10^{-6}	.001	147697	147697	147324	151579
	.003	127706	138992	156651	161844
	.006	99159	142107	149693	157108
	.01	63863	98896	127366	158051
10^{-5}	.001	147486	147486	148548	154624
	.003	128243	139528	157226	161844
	.006	96109	142485	150060	157487
	.01	66770	99548	128003	157995
10^{-4}	.001	150793	150793	151136	160687
	.003	129050	140267	159430	163307
	.006	93738	142828	150374	157646
	.01	63670	97271	126257	158127

Table 4.8. Standard Deviation for Project Set 2, Nonlinear Model.

c_1	c_2	c_3 = 10	13	16	20
10^{-7}	.001	7499	7499	3911	3407
	.003	2840	3109	2494	604
	.006	1486	2095	1847	896
	.01	53.48	1277	1196	1370
10^{-6}	.001	3662	3857	3911	3332
	.003	2735	3102	2494	696
	.006	1640	2250	1630	866
	.01	68.17	940	867	1101
10^{-5}	.001	3727	3727	2661	2555
	.003	2509	2311	1786	554
	.006	1288	1809	1353	670
	.01	42.45	1067	996	1149
10^{-4}	.001	1363	1363	1303	994
	.003	2464	1539	881	319
	.006	992	1442	882	425
	.01	14.7	565	567	612

Table 4.9. Average Dividend for Project Set 1, Nonlinear Model.

c_1	c_2	c_3 = 10	13	16	20
10^{-7}	.001	8324	8325	7329	6524
	.003	9664	8412	6100	5246
	.006	7271	7891	6946	5860
	.01	5022	5914	6133	5936
10^{-6}	.001	7990	7475	7329	6360
	.003	9677	8449	6100	5284
	.006	7500	7931	6981	5890
	.01	5028	5787	6038	5896
10^{-5}	.001	7499	7499	7089	6043
	.003	9582	8285	5999	5226
	.006	7195	7864	6896	5839
	.01	5017	5842	6087	5912
10^{-4}	.001	6958	6958	6626	5405
	.003	9347	7985	5672	5130
	.006	6907	7731	6767	5780
	.01	5006	5639	5923	5827

Table 4.10. Terminal Wealth for Project Set 2, Nonlinear Model.

c_1	c_2	c_3: 10	13	16	20
10^{-7}	.001	149205	147666	155107	160700
	.003	131269	141215	166853	170563
	.006	107681	145064	151041	158581
	.01	63936	98506	121483	160380
10^{-6}	.001	149205	147666	154347	159801
	.003	130840	140845	166699	170200
	.006	107567	145030	151018	156572
	.01	63936	98506	121488	160380
10^{-5}	.001	154091	153098	157293	163116
	.003	131305	141295	167360	170854
	.006	106743	145075	151082	158652
	.01	63936	98261	121295	160436
10^{-4}	.001	157402	158826	160506	169703
	.003	131937	141824	169176	172242
	.006	102537	145411	151392	158904
	.01	63936	97184	120254	160659

Table 4.11. Standard Deviation for Project Set 2, Nonlinear Model.

c_1	c_2	c_3 = 10	13	16	20
10^{-7}	.001	8203	8582	4459	3327
	.003	2901	2932	2275	889
	.006	2241	2803	1843	985
	.01	0	732	689	1412
10^{-6}	.001	8203	8582	4145	3545
	.003	2813	2924	2184	980
	.006	2170	2223	2155	1052
	.01	0	732	689	1412
10^{-5}	.001	4079	4375	3007	2742
	.003	2559	2192	1633	817
	.006	1976	2337	1487	846
	.01	0	672	633	1191
10^{-4}	.001	1683	1491	1688	1103
	.003	2511	1530	781	471
	.006	1371	1596	982	495
	.01	0	432	445	646

Table 4.12. Average Dividend for Project Set 2, Nonlinear Model.

c_1	c_2	c_3: 10	13	16	20
10^{-7}	.001	9814	9046	7339	6358
	.003	9683	8391	5928	5363
	.006	8174	8194	7133	5989
	.01	5000	5658	5673	5982
10^{-6}	.001	9814	9046	7405	6447
	.003	9695	8420	5943	5401
	.006	8174	8194	7149	5996
	.01	5000	5658	5673	5982
10^{-5}	.001	7990	8100	7010	6119
	.003	9603	8256	5844	5333
	.006	8062	8160	7086	5973
	.01	5000	5640	5652	5982
10^{-4}	.001	7423	7039	6527	5450
	.003	9369	7955	5588	5192
	.006	7576	7972	6828	5844
	.01	5000	5539	5549	5864

J-7, in which the responses of G′ and $\bar{D}$ against c_1 are shown, respectively. Figure J-1 consists of nearly horizontal lines, which means that the terminal wealth is almost independent of c_1. A similar thing happens in Figure J-7. We may observe in this figure that there are some lines which have a small negative slope, which implies a decrement in $\bar{D}$ as c_1 is increased. However, the maximum decrement happens to be less than 1,500 and so, in general, we may say that c_1 does not affect $\bar{D}$.

With the exception of two lines, small variations seem to occur in Figure J-4, which shows the standard deviation against c_1. However, the declining tendency of all the lines is clearly seen, showing the general effect of c_1. This tendency is exactly what we expected to obtain: c_1 works against the quadratic differences of dividend payments, which causes the standard deviation for them to be reduced.

After finding that the effect of c_1 is the least important among the parameter effects, the plots of G′ and s against c_2 and c_3 may be presented without the effect of c_1. This reduces the number of lines from 16 to 4, rendering the plots much clearer.

Looking at Figure J-2, the effects of c_2 on G′ are clear. There is a strong tendency to reduce G′ at low values of c_3, while keeping G′ at almost the same level when c_3 is higher. What it means is that the model loses sensitivity to c_2 when there are low "horizon pay-outs."

Parameter c_2 also reduces the standard deviation of the dividend payments, as may be seen in Figure J-5. Note that for $c_3 = 10$, the standard deviation is near zero. All would be fine for the stockholders except for the fact that under those conditions, the dividends are almost D_{min}, and the terminal wealth reaches its lowest value, as it may be seen in Table 4.7 and 4.8.

Regarding $\bar{D}$, parameter c_2 tries to reduce it, as Figure J-8 shows. It seems that for $c_2 = 0.003$ and low values of c_3, the model shows an unexpected behavior: $\bar{D}$ for those values, instead of continually decreasing, becomes higher (note that the same thing happens in project set 2, in Table 4.12). We may conjecture that this region is a region of instability.

Parameter c_3, as in the linear model, is the one which results in the greatest changes. In Figure J-3, it may be seen that G' increases substantially for high values of c_2, and remains more stable at low values of the same parameter. This is well explained given that c_2 is the parameter which assigns the weight to the changes made by c_3 in the objective function.

Regarding s, two different types of behavior due to c_3 are seen in Figure J-6. At high values of c_2, the standard deviation increases its value, while at low values of c_2, it decreases. The same is true for $\bar{D}$.

Regarding the acceptance of projects, the most interesting characteristic of the nonlinear model is that at low values of c_3, the projects which have a life beyond the horizon year are selected in such a way that the terminal wealth consists basically of residual values of those projects. The contrary is true for high values of c_3. This may be seen in Tables H-1 and H-5. In H-1, projects 3, 5, and 8 are accepted (only these projects have post-horizon cash flows), and the money lent at t = 5 has an average of 41,000, while the terminal wealth is 128,000. On the other hand, in H-5 we may see that none of those projects is selected, and all the terminal wealth consists of the money lent at t = 5.

The results obtained using the nonlinear model contain more decision alternatives than the linear case. The major advantage of the nonlinear programming formulation is the great variety of solutions we may choose according to our objectives and desires. However, there are sets of solutions which would normally not be considered, as those for $c_2 = 0.01$ and $c_3 = 10$, in which the company earns little and the stockholders receive the minimum. In other cases, there are solutions which apply to classical situations, as the following:

a) A company which is new in business should keep most of the money it earns and pay minimum dividends. With $c_1 = 1.0 \times 10^{-4}$, $c_2 = 0.003$ and $c_3 = 20$, the company may obtain that type of solution.

b) A company which wants earnings as high as possible, a fair return to its stockholders with the maximum steadiness, should operate near $c_1 = 1.0 \times 10^{-4}$, $c_2 = 0.003$, $c_3 = 13$.

c) A company which prefers to pay the maximum possible dividends with little regard for dividend steadiness or terminal wealth should operate near $c_1 = 1.0 \times 10^{-7}$, $c_2 = 0.003$, and $c_3 = 10$.

A similar analysis may be made for each of the points obtained in the tables. Considerations more extensive may be made to differentiate the advantages and disadvantages between the solutions given in the tables. For example, we have not mentioned (because it is beyond the scope of the present work) considerations about financial ratios, which in most cases, are constrained to specific ranges by company policies. What it means is that there may be situations in which borrowing activities cannot be made without violating such policies, and another solution should be selected with less borrowing or with a different set of selected projects. Example of this are the solutions for $c_1 = 1.0 \times 10^{-4}$, $c_2 = 0.003$, $c_3 = 13$, and $c_1 = 1.0 \times 10^{-7}$, $c_2 = 0.006$, $c_3 = 13$.

Optimality Analysis

The algorithm used to solve our nonlinear model (Beale's method) has some difficulties in solving problems with large coefficients. Apparently, the primal solution is not as strongly affected as the dual. This was corroborated by the author when the optimality conditions were checked. Most of them turned out to be incorrect because of numerical problems. In order to show that the optimality conditions of the nonlinear model presented in Chapter III are correct, the projects in set 1 were scaled (divided by 1000), and a set of parameters selected at random. With the results of this run, every one of the optimality conditions was satisfied.

The set of parameters selected at random were $c_1 = 1.0 \times 10^{-7}$, $c_2 = 0.006$, $c_3 = 20$. These numbers were not scaled to obtain smaller coefficients. Of course, the results are completely different from those of the original problem. The results of the scaled problem are the following:

$x_3 = 1 \quad x_5 = 1 \quad x_6 = 1$

$v_5 = 73.82$

$w_2 = 25 \quad w_3 = 42.52 \quad w_4 = 26.82$

$D_0 = 5 \quad D_1 = 5 \quad D_2 = 5 \quad D_3 = 5 \quad D_4 = 25.4711 \quad D_5 = 5$

Zero otherwise

The corresponding dual variables are:

$(u_M)_0 = 1.0160126 \quad (u_M)_1 = 0.9236478 \quad (u_M)_2 = 0.8389172$

$(u_M)_3 = 0.7619593 \quad (u_M)_4 = 0.6920611 \quad (u_M)_5 = 0.6344829$

$(u_x)_3 = 0.3236176 \quad (u_x)_5 = 0.3595537 \quad (u_x)_6 = 0.0290543$

$(u_D)_0 = -0.018995 \quad (u_D)_1 = -0.0128749 \quad (u_D)_2 = -0.0069495$

$(u_D)_3 = -0.0026542 \quad (u_D)_5 = -0.0040594$

Zero otherwise

First, we may note that the complementary slackness conditions are valid for each of the results. For example, those projects which are not fully accepted have $(u_x)_j = 0$. In addition, $(u_D)_4$ is zero because the dividend payment at $t = 0$ is the only one which is strictly greater than D_{min}.

The relationship (3.108) may be used to check the $(u_M)_t$'s. Using the numerical results, we obtain:

$$\frac{(u_M)_0}{(u_M)_1} = 1.1 \qquad \frac{(u_M)_1}{(u_M)_2} = 1.101 \qquad \frac{(u_M)_2}{(u_M)_3} = 1.101$$

$$\frac{(u_M)_3}{(u_M)_4} = 1.101 \qquad \frac{(u_M)_4}{(u_M)_5} = 1.109$$

All the numbers obtained above are within the range given by (3.108). Note that the second, third, and fourth ratios are at the upper bound, and the fifth at the lower bound: We may see that there is borrowing during the second, third, and fourth periods, and lending during the fifth.

Equation (3.109) may be used to compute the theoretical value of $(u_M)_5$:

$$(u_M)_5 = \frac{2(0.006)}{20(1.095)^5}\left[\frac{20}{2(0.006)} - \frac{143.47985}{20(1.095)^5} + \frac{38.43775137}{6}\right]$$

$$= 0.63593245 \simeq 0.6344829$$

The theoretical value of each of $(u_x)_j$ is given by (3.110). We may check the dual variables for projects 3, 5 and 6, which are fully accepted:

$$(u_x)_3 = -20(0.9236478) - 10(0.8389172) + 5(0.7619593)$$
$$-5(0.6920611) + 10(0.6345) + 32.2512(0.6345)$$
$$= 0.29575 \simeq 0.3236716$$

$$(u_x)_5 = -15(1.0160126 + 0.9236478 + 0.8389172 + 0.7619593)$$
$$+20(0.6920611) + 25(0.6344829) + 37.40365(0.6344829)$$
$$= 0.32721 \approx 0.359553$$

$$(u_x)_6 = -50(1.0160126) + 20(0.7619593) + 25(0.6920611)$$
$$+ 28.825(0.6344829)$$
$$= 0.029053 \simeq 0.0290543$$

Equation (3.110) also works for projects which are not selected. Let's check project 10, for example:

$$(u_x)_{10} = -10(1.0160216) - 10(0.9236478) + 35(0.8389172)$$
$$-13.09(0.7619593) + 0 = -0.00855 \simeq 0$$

The dual variables associated with the dividend policy (3.52) may be obtained from (3.112), (3.113), and (3.114).

$$(u_D)_0 \leq 1.0160126 - \frac{2(0.006)}{6} \left[\frac{143.47985}{20(1.095)^5} - \frac{1}{6}(38.4377)\right]$$

$$+ 2(10^{-7})(5 + 5) - 1 = 0.019713 \geq -0.018995$$

$$(u_D)_3 \leq 0.7619593 - \frac{2(0.006)}{6} \left[\frac{143.47985}{20(1.095)^5} - \frac{1}{6}(38.4377)\right]$$

$$+ 2(10^{-7})(5 + 2*5 + 25.4711) - \frac{1}{(1.095)^3}$$

$$= 0.0040119 \geq -0.0026452$$

$$(u_D)_5 \leq 0.6344829 \ (1 + \frac{20}{6}) + 2(10^{-7})(25.4711 + 5)$$

$$- \frac{1}{(1.095)^5} (1 + \frac{20}{6}) = -0.003190 \geq -0.00406$$

At year t = 4, the company pays dividends higher than D_{min}, so the dual variable $(u_D)_4$ can be calculated from (3.118):

$$(u_D)_4 = 0.6920611 - \frac{2(0.006)}{6(1.095)^5} (-1.849165) + 2(10^{-7})$$

$$(5+2*25.4711 + 5) - \frac{1}{(1.095)^4} = -0.009346 \approx 0$$

Note from the way in which the original primal problem is presented, that the dual variables $(u_D)_t$ should be non-positive which is exactly what we obtained above. This negative sign means that the value of the objective function will decrease by $(u_D)_t$ if the right hand side of (3.52) is increased by one. Obviously, $(u_D)_4$ is zero because if D_{min} at t = 4 were increased by one, the company will still pay 25.4711 and no changes in the objective function will occur.

As mentioned before, project set 2 was also solved with the nonlinear model. As evidenced by Tables 4.10, 4.11 and 4.12, as well as Appendix I, the results follow the same pattern as those of set 1. What it means is that the nonlinear model does not have the lack of sensitivity to the rates of return of the projects which the linear model showed.

Because of this fact no further comments are made on the set 2 results given that those made for set 1 hold similarly.

Summary

We have presented the results of our two models using two sets of projects. Their characteristics have been discussed briefly. We have shown that c_3 is the most important parameter with respect to the effects on average dividend, standard deviation, and terminal wealth. The optimality conditions presented in Chapter III were checked, and the numerical results agree for the linear model. A non-linear problem had to be rescaled to demonstrate the optimality conditions, becuase of the numerical problems associated with Beale's method. The results obtained from several runs provide the information which a decision-maker needs for investment planning. In the next and concluding chapter, the advantages and disadvantages of the models will be discussed.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Objectives of the Research Achieved

As stated in Chapter I, the main objectives of this thesis were to develop a capital budgeting model which contains dividend variables. A desired feature of such a model is that the dividends are free to fluctuate about an average level, but that the variability of dividends be somehow controlled. In addition, it is desired to maintain some relationship between the terminal wealth and the dividends paid.

We have presented two models, one linear and one nonlinear, which deal with the issues discussed above. The approaches used are quite different from each but their objectives are the same. The linear model has in its objective function a term representing the slope of the dividend payment series. In addition, a term is included in order to control the difference between terminal wealth and dividends. On the other hand, the nonlinear model includes in its objective function the squared differences between subsequent dividend payments, and the squared difference between terminal wealth and average dividend.

Through the use of duality theory, the optimality conditions for the two models are examined. These conditions were useful in finding the ranges of the parameters contained in the models. In addition, they provided economic interpretations.

Two numerical examples were used to demonstrate that the models achieve the objectives for which they were designed. A variety of results were obtained by using a factorial design for the parameter values.

Conclusions

We have discussed the features as well as some drawbacks of the models in previous chapters. The major points are repeated here:

The linear model has several drawbacks which lead to some undesirable characteristics in the numerical results. One drawback comes from the use of the slope estimator of the dividend payment series. Its use assumes that the dividends can be regressed as a straight line against time, an assumption which forces the dividend stream to assume some peculiar patterns. It turns out that the slope is always made zero but that the fluctuations in dividends are not controlled.

Another drawback stems from the linear model itself. As is shown in (3.45), some coefficients of the dividends and terminal wealth can be factored out. As a result, the slope term and the term representing the difference between terminal wealth and average dividend, vanish from the objective function. This can also be seen in the lower bound on the dual objective function (3.44), which is independent of c_1 and c_2. However, c_1 and c_2 do affect the dividend pattern, and so provide a large number of solutions with similar objective function values. A similar thing happens with c_1 in the nonlinear model. However, the insensitivity of c_1 here is not as persistent as in the linear case.

In the optimality conditions for both models, several relationships which belong to classical capital budgeting models were found to hold also for our models. Such is the case of (3.26) and (3.108), which represent the ratios of subsequent dual variables associated with the budget constraints. On the other hand, while in the basic horizon model, the dual variable for the last budget constraint, $(u_M)_T$, is equal to one, for our nonlinear model, it is a function of the parameters c_2 and c_3, the terminal wealth, the dividends, and the discount rate. This $(u_M)_T$ represents the increase of the objective function value if the budget for $t = T$ were increased by one unit.

In general, we may conclude that the nonlinear model works much better than the linear one. The nonlinear model considers adequately the variability of dividends and the maintenance of terminal wealth. It provides results for any levels of the parameters, and it does not have infeasible regions for parameter values. In addition, it has a high sensitivity to changes in the parameter values. This sensitivity has been an important consideration since the start of this research. The nonlinear model works well in selecting different sets of projects and paying different streams of dividends for different parameter values with the second set of projects, while the linear model selects the same two projects for every feasible combination of parameter values. The conclusion which may be drawn here is that the linear model reacts strongly to projects with different rates of return, selecting those with the highest. On the other hand, the nonlinear model selects the set of projects considering the weights given by c_1, c_2, and c_3 to the utility terms in the objective function.

Recommendations

Further research in this area should be focused on the use of nonlinear programming. The advantages which it offers have been shown in this thesis. However, Beale's algorithm presents problems in solving nonlinear problems with a large range of coefficient values, and this is probably true for most nonlinear algorithms. So, an important recommendation for the practitioners of capital budgeting is that in dealing with the nonlinear model presented here, or any similar model, one should rescale the cash flows of the projects.

The Beale's algorithm used is quite old. It was developed in the middle of the sixties. The reader might locate a newer algorithm or code which does not present the drawbacks mentioned above.

The linear model presented in this thesis has many drawbacks, and its results do not fulfill completely the objective of obtaining streams of dividends with high stability. The use of the estimator of the dividends' slope does not work as we expected. However, this does not imply that the use of linear programming cannot achieve a stream of dividends with low variability. It is left for further research to formulate linear models for doing this, but a good approach would be to minimize in (3.1) not the dividends' slope, but the dividends' deviations from the average dividend $\bar{D}$, as defined in (4.1). In order to do that, we might have to define nonnegative deviations as follows:

d_t^+ deviation of D_t from $\bar{D}$, when $D_t > \bar{D}$

d_t^- deviation of D_t from $\bar{D}$, when $D_t < \bar{D}$

In addition, more constraints should be established:

$$D_t \leq \bar{D} + d_t^+ \qquad t = 0, 1, \ldots, T \qquad (5.1)$$

$$D_t \geq \bar{D} - d_t^- \qquad t = 0, 1, \ldots, T \qquad (5.2)$$

$$d_t^+, \; d_t^- \geq 0 \qquad t = 0, 1, \ldots, T \qquad (5.3)$$

And (3.1) can be rewritten as follows:

Maximize:

$$\sum_{t=0}^{T} \frac{D_t}{(1+k)^t} + [\sum_{j=0}^{n} \hat{a}_j x_j + v_t - w_t]/(1+k)^T - c_1[\sum_{t=0}^{T} (d_t^+ + d_t^-)]$$

$$-c_2[(\sum_{j=0}^{n} \hat{a}_j x_j + v_T - w_T)/(1+k)^T c_3 - \frac{1}{T+1} \sum_{t=0}^{T} \frac{D_t}{(1+k)^t}] \tag{5.4}$$

The number of constraints of this model is greater than the model in Chapter III, so the optimality analysis might be more complicated. In addition, some iterations may be needed to establish an appropriate value of $\bar{D}$.

Finally, the next step in improving the present work, is to use integer programming for solving our nonlinear model. This will add more realism to the results obtained so far.

APPENDIX A

SUBMIT File Code of the Linear Model

```
/JOB
/NOSEQ
JOB,T200.
USER.
GET,MODEL1.
GET,DATA.
HEADING.1 SET 1.
HEADING.1 C1 8. C2 1.5  C3 2.
FTN,I=MODEL1,L=0,ER.
LGO,DATA,OUT.
R,OUT.
GET,EZLP/UN=TE306AC.
CALL,EZLP.
/EOR
RUN,OUT
LIST
USE PRTMAL
END
/EOR
```

APPENDIX B

FORTRAN Code of the Linear Model

```
      PROGRAM MODEL1(INPUT,OUTPUT)
      DIMENSION A(20,20),CD1(20),CD2(20),CD3(20),CD(20),CD4(20),
     XCD5(20),CX1(20),CX2(20),CX(20),CX3(20),BM(20),RB(20),
     XRL(20),XLEND(20),XBORR(20),NS(50),Y(20),NE(20),JJ(20),
     XB(20),C(20,20),NSS(20,20),NP(20),NNE(20,20),NNP(20),JJJ(20,20)
C     ******************************************************************
C     *
C     *    PROGRAM WRITTEN BY PATRICIO G. MURGA
C     *    SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING
C     *    GEORGIA INSTITUTE OF TECHNOLOGY
C     *
C     *    COPYRIGTH BY P.G. MURGA
C     *
C     *    MOST RECENT VERSION OF THE PROGRAM : 8/30/78
C     *    CREATED : 5/15/78
C     *    LOCATED IN MODEL1,IE302AJ
C     *
C     *
C     *    THIS PROGRAM CALCULATES THE COEFFICIENTS OF THE LINEAR
C     *    MODEL AND ARRANGE THEM IN A CONVENIENT WAY TO INPUT THE
C     *    DATA TO EZLP.
C     *
C     *
C     *    DESCRIPTION OF DATA INPUT:
C     *
C     *        N       NUMBER OF PROJECTS
C     *        NT      LENGTH OF THE HORIZON
C     *        DMIN    MINIMUM DIVIDENDS PAID PER YEAR
C     *        M       MAXIMUM PROJECT LIFE
C     *        C1      VALUE OF THE PARAMETER ASSIGNED TO THE DIVIDEND
C     *                SLOPE
C     *        C2      VALUE OF THE PARAMETER ASSIGNED TO THE DIFFERENCE
C     *                BETWEEN TERMINAL WEALTH AND DIVIDENDS
C     *        C3      VALUE OF THE PARAMETER INVOLVED IN THE HORIZON
C     *                PAY-OUT RATIO
C     *        XK      STOCKHOLDERS' DISCOUNT RATE
C     *        A       MATRIX CONTAINING THE CASH FLOWS OF THE PROJECTS
C     *        RB      VECTOR CONTAINING THE BORROWING RATES FOR THE
C     *                HORIZON
C     *        RL      VECTOR CONTAINING THE LENDING RATES FOR THE
C     *                HORIZON
C     *        BM      VECTOR CONTAINING THE BUDGET CONSTRAINTS FOR THE
C     *                HORIZON
C     *
C     *
C     *
C     ******************************************************************
      DATA M/10/
      DATA C1/8./
      DATA C2/1.5/
      DATA C3/2./
      NTT=NT+1
      DATA XK/0.095/
      M=M+1
      READ 100,((A(I,J),J=1,M),I=1,N)
      READ 101,(RB(I),I=1,NTT)
      READ 101,(RL(I),I=1,NTT)
      READ 102,(BM(I),I=1,NTT)
  100 FORMAT(7F10.0)
  101 FORMAT(6F5.3)
  102 FORMAT(6F7.0)
```

```
C
C     DIVIDENDS COEFFICIENTS COMPUTATION
C
      TT=NT
      DO 2  IT=1,NTT
      XLEND(IT)=1+RL(IT)
      XBORR(IT)=1+RB(IT)
      IF(IT.GT.1)  GO TO 3
      LT2=0
      DO 4  ITT=1,NTT
      LT2=LT2+(ITT-1)**2
    4 CONTINUE
      T2=LT2
    3 CONTINUE
      T=IT-1
      CD1(IT)=1/(1+XK)**T
      CD4(IT)=(T-TT/2)/(T2-(TT**2)*(TT+1)/4)
      CD3(IT)=C2/((TT+1)*(1+XK)**T)
      CD2(IT)=CD4(IT)*C1
      CD(IT)=CD1(IT)-CD2(IT)+CD3(IT)
      CD5(IT)=CD3(IT)/C2
    2 CONTINUE
C
C     PROJECTS COEFFICIENTS COMPUTATION
C
      DO 1  J=1,N
      CX1(J)=0
      NTTT=NTT+1
      DO 5  I=NTTT,M
      T=I-NTT
      CX1(J)=CX1(J)+A(J,I)/(1+XK)**T
    5 CONTINUE
      CX1(J)=CX1(J)/((1+XK)**TT)
      CX2(J)=CX1(J)*C2/C3
      CX(J)=CX1(J)-CX2(J)
      CX3(J)=CX1(J)/C3
    1 CONTINUE
C
C     COMPUTATION OF THE COEFFICIENTS RELATED TO BORROWING AND
C     LENDING
C
      CV1=1/(1+XK)**TT
      CV3=CV1/C3
      CV2=CV3*C2
      CV=CV1-CV2
      CW=CV
      CW3=CV3
      DO 6  I=1,NTT
      Y(I)=ABS(CD(I))
      IF(CD(I).GE.0)  GO TO 7
      NS(I)=1H-
      GO TO 8
    7 NS(I)=1H+
      IF(I.NE.1)  GO TO 8
      Y(1)=CD(1)
      NS(1)=1H
    8 CONTINUE
      NE(I)=1H
    6 CONTINUE
      NE(NTT)=1H&
      DO 9  I=1,NTT
```

```
      JJ(I)=I-1
    9 CONTINUE
C
C     PRINT OBJECTIVE FUNCTION
C
      PRINT 10
   10 FORMAT(1X,"MAX:&")
      IF(NTT.GT.5) GO TO 200
      PRINT 11,(NS(I),Y(I),JJ(I),NE(I),I=1,NTT)
      GO TO 201
   11 FORMAT(10(1X,A1,F8.3,"D",I1,A1))
  200 NE(5)=1H&
      PRINT 11,(NS(I),Y(I),JJ(I),NE(I),I=1,5)
      PRINT 11,(NS(I),Y(I),JJ(I),NE(I),I=6,NTT)
  201 CONTINUE
      NUM=0
      J=0
      NUM2=0
      DO 12 I=1,N
      IF(CX(I).EQ.0) GO TO 13
      J=J+1
      IF(J.GE.10) GO TO 17
      NUM=NUM+1
      GO TO 18
   17 NUM2=NUM2+1
   18 CONTINUE
      IF(CX(I).GE.0) GO TO 14
      B(J)=ABS(CX(I))
      NS(J)=1H-
      GO TO 15
   14 B(J)=CX(I)
      NS(J)=1H+
   15 JJ(J)=I
   13 CONTINUE
   12 CONTINUE
      NE(NUM)=1H&
      IF(NUM.GT.5) GO TO 202
      PRINT 16,(NS(I),B(I),JJ(I),NE(I),I=1,NUM)
      GO TO 203
   16 FORMAT(9(1X,A1,F9.0,"X",I1,A1))
  202 NE(5)=1H&
      PRINT 16,(NS(I),B(I),JJ(I),NE(I),I=1,5)
      PRINT 16,(NS(I),B(I),JJ(I),NE(I),I=6,NUM)
  203 CONTINUE
      IF(NUM2.LE.0) GO TO 19
      K=NUM+1
      NUM2=NUM+NUM2
      NE(NUM2)=1H&
      PRINT 20,(NS(I),B(I),JJ(I),NE(I),I=K,NUM2)
   20 FORMAT(9(1X,A1,F9.0,"X",I2,A1))
   19 JJ(1)=NT
      IF(CV) 21,22,23
   21 CWW=ABS(CW)
      PRINT 24,CV,JJ(1),CWW,JJ(1)
   24 FORMAT(1X,F9.4,"V",I1,"+",F9.4,"W",I1)
      GO TO 22
   23 PRINT 25,CV,JJ(1),CW,JJ(1)
   25 FORMAT(1X,"+",F9.4,"V",I1,"-",F9.4,"W",I1)
   22 CONTINUE
      DO 26 J=1,NTT
      NP(J)=0
```

```
      NNP(J)=0
      K=0
      DO 27 I=1,N
      IF(A(I,J).EQ.0) GO TO 28
      K=K+1
      IF(K.NE.1) GO TO 31
      C(K,J)=-A(I,J)
      NSS(K,J)=1H
      GO TO 30
   31 C(K,J)=ABS(A(I,J))
      IF(A(I,J).GE.0) GO TO 29
      NSS(K,J)=1H+
      GO TO 30
   29 NSS(K,J)=1H-
   30 CONTINUE
      NNE(K,J)=1H
      JJJ(K,J)=I
      IF(I.GE.10) GO TO 35
      NP(J)=NP(J)+1
      GO TO 28
   35 NNP(J)=NNP(J)+1
   28 CONTINUE
   27 CONTINUE
      K=NP(J)
      NNE(K,J)=1H$
   26 CONTINUE
C
C     PRINT CONSTRAINTS
C
      PRINT 34
   34 FORMAT(1X,"ST:&")
      L=NP(1)
      IF(L.GT.5) GO TO 204
      PRINT 36,(NSS(I,1),C(I,1),JJJ(I,1),NNE(I,1),I=1,L)
      GO TO 205
   36 FORMAT(9(1X,A1,F8.0,"X",I1,A1))
  204 NNE(5,1)=1H$
      PRINT 36,(NSS(I,1),C(I,1),JJJ(I,1),NNE(I,1),I=1,5)
      PRINT 36,(NSS(I,1),C(I,1),JJJ(I,1),NNE(I,1),I=6,L)
  205 CONTINUE
      IF(NNP(1).EQ.0) GO TO 37
      K=NP(1)+1
      NNP(1)=NNP(1)+NP(1)
      L=NNP(1)
      NNE(L,1)=1H$
      PRINT 38,(NSS(I,1),C(I,1),JJJ(I,1),NNE(I,1),I=K,L)
   38 FORMAT(6(1X,A1,F8.0,"X",I2,A1))
   37 CONTINUE
      PRINT 39,BM(1)
   39 FORMAT(1X,"+V0-W0+D0<=",F7.0)
      DO 40 J=2,NTT
      PRINT 41
   41 FORMAT(1X,"AND:&")
      L=NP(J)
      IF(L.GT.5) GO TO 206
      PRINT 42,(NSS(I,J),C(I,J),JJJ(I,J),NNE(I,J),I=1,L)
      GO TO 207
   42 FORMAT(9(1X,A1,F8.0,"X",I1,A1))
  206 NNE(5,J)=1H$
      PRINT 42,(NSS(I,J),C(I,J),JJJ(I,J),NNE(I,J),I=1,5)
      PRINT 42,(NSS(I,J),C(I,J),JJJ(I,J),NNE(I,J),I=6,L)
```

```
207  CONTINUE
     IF(NNP(J).EQ.0) GO TO 43
     K=NP(J)+1
     NNP(J)=NP(J)+NNP(J)
     L=NNP(J)
     NNE(L,J)=1H&
     PRINT 44,(NSS(I,J),C(I,J),JJJ(I,J),NNE(I,J),I=K,L)
 44  FORMAT(5(1X,A1,F8.0,"X",I2,A1))
 43  CONTINUE
     K=J-1
     L=K-1
     PRINT 45,K,K,K,XLEND(J),L,XBORR(J),L,BM(J)
 45  FORMAT(1X,"+V",I1,"-W",I1,"+D",I1,"-",F5.3,"V",I1,"+",F5.3,
    X"W",I1,"<=",F7.0)
 40  CONTINUE
     DO 46 I=1,NTT
     IF(CD4(I).GE.0) GO TO 47
     IF(I.NE.1) GO TO 99
     B(I)=CD4(I)
     GO TO 48
 99  B(I)=ABS(CD4(I))
     NS(I)=1H-
     GO TO 48
 47  B(I)=CD4(I)
     NS(I)=1H+
 48  CONTINUE
     JJ(I)=I-1
     NE(I)=1H
 46  CONTINUE
     NS(1)=1H
     NE(NTT)=3H>=0
     PRINT 49
 49  FORMAT(1X,"AND:&")
     IF(NTT.GT.5) GO TO 208
     PRINT 50,(NS(I),B(I),JJ(I),NE(I),I=1,NTT)
     GO TO 209
 50  FORMAT(10(1X,A1,F6.4,"D",I1,A3))
208  NE(5)=1H&
     PRINT 50,(NS(I),B(I),JJ(I),NE(I),I=1,5)
     PRINT 50,(NS(I),B(I),JJ(I),NE(I),I=6,NTT)
209  CONTINUE
     LL=2H>=
     DO 51 J=1,NTT
     PRINT 52,JJ(J),LL,DMIN
 52  FORMAT(1X,"AND: D",I1,A2,F7.0)
 51. CONTINUE
     NUM=0
     NUM2=0
     J=0
     DO 53 I=1,N
     IF(CX3(I).EQ.0) GO TO 53
     J=J+1
     IF(J.GE.10) NUM2=NUM2+1
     NUM=NUM+1
     IF(CX3(I)) 54,54,55
 55  B(J)=CX3(I)
     NS(J)=1H+
     GO TO 56
 54  B(J)=ABS(CX3(J))
     NS(J)=1H-
 56  CONTINUE
```

```
      IF(J.NE.1) GO TO 57
      B(J)=CX3(I)
      NS(J)=1H
  57  CONTINUE
      NE(J)=1H
      JJ(J)=I
  53  CONTINUE
      NE(NUM)=1H&
      PRINT 58
  58  FORMAT(1X,"AND# &")
      IF(NUM.GT.4) GO TO 210
      PRINT 59,(NS(I),B(I),JJ(I),NE(I),I=1,NUM)
      GO TO 211
  59  FORMAT(8(1X,A1,F11.3,"X",I1,A1))
 210  NE(4)=1H&
      PRINT 59,(NS(I),B(I),JJ(I),NE(I),I=1,4)
      PRINT 59,(NS(I),B(I),JJ(I),NE(I),I=5,NUM)
 211  CONTINUE
      IF(NUM2.EQ.0) GO TO 60
      K=NUM+1
      NUM2=NUM+NUM2
      NE(NUM2)=1H%
      PRINT 61,(NS(I),B(I),JJ(I),NE(I),I=K,NUM2)
  61  FORMAT(1X,A1,F11.3,"X",I2,A1)
  60  CONTINUE
      JJ(1)=NT
      PRINT 62,CV3,JJ(1),CW3,JJ(1),NE(NUM)
  62  FORMAT(1X,"+",F10.4,"V",I1,"-",F10.4,"W",I1,A1)
      DO 63 I=1,NTT
      NS(I)=1H-
      NE(I)=1H
      JJ(I)=I-1
  63  CONTINUE
      NE(NTT)=3H>=0
      IF(NTT.GT.4) GO TO 212
      PRINT 64,(NS(I),CD5(I),JJ(I),NE(I),I=1,NTT)
      GO TO 213
  64  FORMAT(7(1X,A1,F10.4,"D",I1,A3))
 212  NE(4)=1H&
      PRINT 11,(NS(I),CD5(I),JJ(I),NE(I),I=1,4)
      PRINT 64,(NS(I),CD5(I),JJ(I),NE(I),I=5,NTT)
 213  CONTINUE
      DO 65 J=1,N
      IF(J.GE.10) GO TO 67
      PRINT 68,J
  68  FORMAT(1X,"AND# X",I1,"<=1")
      GO TO 69
  67  CONTINUE
      PRINT 70,J
  70  FORMAT(1X,"AND# X",I2,"<=1")
  69  CONTINUE
  65  CONTINUE
      PRINT 71
  71  FORMAT(1X,"AND# ALL VARS>=0")
      STOP
      END
```

APPENDIX C

SUBMIT File Code of the Nonlinear Model

```
/JOB
/NOSEQ
JOB,T200.
USER.
GET,MODEL2.
GET,DATA.
FTN,I=MODEL2,L=0,ER.
LGO,DATA,DATFIL.
R,DATFIL.
ATTACH,MPOS/UN=LIBRARY.
MPOS.
/EOR
TITLE
C1 0.0001 C2 0.006 C3 20.
REALE
VARIABLES
X1 TO X28
PACKED
MAXIMIZE
CONSTRAINTS 22
++++++------++++++++++
FORMAT
(2I10,F20.5)
READ DATFIL
CHECK
OPTIMIZE
STOP
/EOR
```

APPENDIX D

FORTRAN Code of the Nonlinear Model

```
      PROGRAM MODEL2(INPUT,OUTPUT)
      DIMENSION A(50,50),RB(50),RL(50),BM(50),Q(50,50),T(50,50),
     1C(50),PWHX(50)
C     ******************************************************************
C     *
C     *     PROGRAM WRITTEN BY PATRICIO G. MURGA
C     *     SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING
C     *     GEORGIA INSTITUTE OF TECHNOLOGY
C     *
C     *     COPYRIGTH BY P.G. MURGA
C     *
C     *     MOST RECENT VERSION OF THE PROGRAM : 8/30/78
C     *     CREATED : 5/15/78
C     *     LOCATED IN MODEL2,IE302AJ
C     *
C     *
C     *     THIS PROGRAM CALCULATES THE COEFFICIENTS OF THE NONLINEAR
C     *     MODEL AND ARRANGE THEM IN A CONVENIENT WAY TO INPUT THE
C     *     DATA TO MPOS.
C     *
C     *
C     *     DESCRIPITON OF DATA INPUT:
C     *
C     *         N        NUMBER OF PROJECTS
C     *         NT       LENGTH OF THE HORIZON
C     *         DMIN     MINIMUM DIVIDENDS PAID PER YEAR
C     *         M        MAXIMUM PROJECT LIFE
C     *         C1       VALUE OF THE PARAMETER ASSIGNED TO THE QUADRATIC
C     *                  DIFFERENCES OF THE DIVIDENDS
C     *         C2       VALUE OF THE PARAMETER ASSIGNED TO THE DIFFERENCE
C     *                  BETWEEN TERMINAL WEALTH AND DIVIDENDS
C     *         C3       VALUE OF THE PARAMETER INVOLVED IN THE HORIZON
C     *                  PAY-OUT RATIO
C     *         XK       STOCKHOLDERS' DISCOUNT RATE
C     *         A        MATRIX CONTAINING THE CASH FLOWS OF THE PROJECTS
C     *         RB       VECTOR CONTAINING THE BORROWING RATES FOR THE
C     *                  HORIZON
C     *         RL       VECTOR CONTAINING THE LENDING RATES FOR THE
C     *                  HORIZON
C     *         BM       VECTOR CONTAINING THE BUDGET CONSTRAINTS FOR THE
C     *                  HORIZON
C     *
C     *
C     *
C     ******************************************************************
      DATA XK/0.095/
      DATA C1,C2,C3/0.0001,6.,20./
      DATA M/10/
      DATA N,NT,DMIN/10,5,5.000/
      M=M+1
      NTT=NT+1
      READ 1,((A(I,J),J=1,M),I=1,N)
      READ 2,(RB(J),J=1,NTT)
      READ 2,(RL(J),J=1,NTT)
      READ 3,(BM(J),J=1,NTT)
    1 FORMAT(7F10.0)
    2 FORMAT(6F5.3)
    3 FORMAT(6F7.0)
      NITT=NTT+1
```

```
C
C     COMPUTATION OF THE POST-T CASH FLOWS PRESENT WORTH
C
      DO 10 I=1,N
      PWHX(I)=0.
      DO 11 J=NTTT,M
      TT=J-NTT
   11 PWHX(I)=PWHX(I)+A(I,J)/(1+XK)**TT
   10 CONTINUE
C
C     COMPUTATION OF THE QUADRATIC TERMS COEFFICIENTS OF THE
C     OBJECTIVE FUNCTION
C
      DO 12 J=1,N
      DO 13 I=1,N
      Q(I,J)=-C2*PWHX(I)*PWHX(J)/(C3*(1+XK)**NT)**2
   13 CONTINUE
      N1=N+1
      N2=N+NTT
      DO 14 I=N1,N2
      Q(I,J)=0.
   14 Q(I,J)=Q(J,I)
      Q(N2,J)=-C2*PWHX(J)/(C3*(1+XK)**NT)**2
      Q(J,N2)=Q(N2,J)
      N3=N2+1
      N4=2*NTT+N
      DO 15 I=N3,N4
      Q(I,J)=0.
   15 Q(J,I)=Q(I,J)
      Q(N4,J)=PWHX(J)*C2/(C3*(1+XK)**NT)**2
      Q(J,N4)=Q(N4,J)
      N5=N4+1
      N6=3*NTT+N
      DO 16 I=N5,N6
      TT=I-N5
      Q(I,J)=PWHX(J)*C2/(C3*(NT+1)*((1+XK)**(NT+TT)))
   16 Q(J,I)=Q(I,J)
   12 CONTINUE
      DO 17 J=N1,N2
      DO 18 I=N1,N2
   18 Q(I,J)=0.
      Q(N2,N2)=-C2/(C3*(1+XK)**NT)**2
      DO 19 I=N3,N4
      Q(I,J)=0.
   19 Q(J,I)=0.
      Q(N4,N2)=C2/(C3*(1+XK)**NT)**2
      Q(N2,N4)=Q(N4,N2)
      DO 20 I=N5,N6
      TT=I-N5
      Q(I,J)=0.
      Q(I,N2)=C2/(C3*((1+XK)**(NT+TT))*(NT+1))
   20 Q(J,I)=Q(I,J)
   17 CONTINUE
      DO 21 J=N3,N4
      DO 22 I=N3,N4
   22 Q(I,J)=0.
      Q(N4,N4)=-C2/(C3*(1+XK)**NT)**2
      DO 23 I=N5,N6
      TT=I-N5
      Q(I,J)=0.
      Q(I,N4)=-C2/((NT+1)*C3*((1+XK)**NT)*((1+XK)**TT))
```

```
23  Q(J,I)=Q(I,J)
21  CONTINUE
    DO 24 J=N5,N6
    DO 25 I=N5,N6
    TI=I-N5
    TJ=J-N5
    IF(J.EQ.I) GO TO 26
    Q(I,J)=-C2/(((NT+1)**2)*((1+XK)**(TI+TJ)))
    GO TO 25
26  Q(I,J)=-C2/(((NT+1)**2)*((1+XK)**(TI+TJ)))-2*C1
25  CONTINUE
24  CONTINUE
    Q(N5,N5)=-C2/(NT+1)**2 -C1
    Q(N6,N6)=-C2/((NT+1)*((1+XK)**NT)**2) -C1
    DO 27 J=N5,N6
    DO 28 I=N5,N6
    TI=I-N5
    TJ=J-N5
    IF(J.NE.I) GO TO 28
    IF(J.EQ.N6) GO TO 28
    JJ=J+1
    TJJ=TJ+1
    Q(JJ,J)=-C2/(((NT+1)**2)*((1+XK)**(TI+TJJ)))+2*C1
    Q(J,JJ)=Q(JJ,J)
28  CONTINUE
27  CONTINUE
C
C
C   COMPUTATION OF THE LINEAR TERMS' COEFFICIENTS OF THE OBJECTIVE
C   FUNCTION
C
    DO 29 J=1,N
29  C(J)=PWHX(J)/(1+XK)**NT
    DO 30 J=N1,N4
30  C(J)=0.
    C(N3)=1/(1+XK)**NT
    C(N4)=-1/(1+XK)**NT
    DO 31 J=N5,N6
    TT=J-N5
31  C(J)=1/(1+XK)**TT
C
C
C   T(I,J) IS A MATRIX OF THE COEFFICIENTS OF THE CONSTRAINTS'
C   VARIABLES
C
    DO 32 I=1,NTT
    DO 33 J=1,N
33  T(I,J)=-A(J,I)
    IF(I.EQ.1) GO TO 34
    DO 35 J=N1,N2
    IF(J.EQ.I+N) GO TO 36
    T(I,J)=0.
    GO TO 35
36  T(I,J)=1.
    JM1=J-1
    T(I,JM1)=-(1+RL(I))
35  CONTINUE
    DO 37 J=N3,N4
    IF(I+N+NTT.EQ.J) GO TO 38
    T(I,J)=0.
    GO TO 37
38  T(I,J)=-1.
    JM1=J-1
```

```
      T(I,JM1)=1+RB(I)
   37 CONTINUE
      GO TO 100
   34 CONTINUE
      DO 101 J=N1,N4
  101 T(I,J)=0.
      T(1,N1)=1.
      T(1,N3)=-1
  100 CONTINUE
      DO 39 J=N5,N6
      IF(J.EQ.I+2*NTT+N) GO TO 40
      T(I,J)=0.
      GO TO 39
   40 T(I,J)=1.
   39 CONTINUE
   32 CONTINUE
      N7=NTT+1
      N8=2*NTT
      DO 41 I=N7,N8
      DO 42 J=1,N4
   42 T(I,J)=0.
      DO 43 J=N5,N6
      IF(J.EQ.I+N+NTT) GO TO 44
      T(I,J)=0.
      GO TO 43
   44 T(I,J)=1.
   43 CONTINUE
   41 CONTINUE
      N9=N8+1
      N10=2*NTT+N
      DO 45 I=N9,N10
      DO 46 J=1,N
      IF(J.EQ.I-2*NTT) GO TO 47
      T(I,J)=0.
      GO TO 46
   47 T(I,J)=1.
   46 CONTINUE
      DO 48 J=N1,N6
   48 T(I,J)=0.
   45 CONTINUE
C
C     DEFINE RIGTH HAND SIDE
C
      DO 49 I=1,NTT
      NE1=N6+1
   49 T(I,NE1)=BM(I)
      DO 50 I=N7,N8
   50 T(I,NE1)=DMIN
      DO 51 I=N9,N10
   51 T(I,NE1)=1.
      DO 60 I=1,N5
      DO 61 J=1,N6
      IF(Q(I,J).EQ.0.) GO TO 61
      II=-I
      JJ=-J
      PRINT 62,II,JJ,Q(I,J)
   62 FORMAT(2I10,F20.5)
   61 CONTINUE
   60 CONTINUE
      DO 63 J=1,N6
```

```
      IF(C(J).EQ.0) GO TO 63
      J0=0
      PRINT 62,J0,J,C(J)
   63 CONTINUE
      DO 64 I=1,N10
      DO 65 J=1,N6
      IF(T(I,J).EQ.0) GO TO 65
      PRINT 62,I,J,T(I,J)
   65 CONTINUE
   64 CONTINUE
      DO 66 I=1,N10
      I0=0
      PRINT 62,I,I0,T(I,N61)
   66 CONTINUE
      STOP
      END
```

APPENDIX E

Selected Results of the Linear Model (Set 1)

TABLE E-1

LINEAR MODEL

SET 1

VALUE OF C2 .5
VALUE OF C3 1.0

	VALUE OF C1			
VARIABLE	0.5	2	8	10
X1	0.000	0.000	0.000	0.000
X2	0.000	0.000	0.000	0.000
X3	1.000	1.000	1.000	1.000
X4	1.000	1.000	.794	.478
X5	.617	.617	.745	.711
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	.525	.525	1.000	1.000
X10	1.000	1.000	0.000	1.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	0.	0.	0.	0.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	66780.	66780.	75650.	73300.
W4	56180.	56180.	63400.	61480.
W5	26420.	26420.	31190.	29920.
D0	5000.	5000.	5000.	5000.
D1	10550.	10550.	30730.	15700.
D2	47700.	47700.	7375.	37480.
D3	64430.	64430.	84480.	69540.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	28924.	28924.	28911.	28921.
AVERAGE DIVIDEND	22946.	22946.	22930.	22953.
STD. DEVIATION	26281.	26281.	31798.	26080.

TABLE E-2

LINEAR MODEL

SET 1

VALUE OF C2 .5
VALUE OF C3 2.0

	VALUE OF C1			
VARIABLE	0.5	2	8	10
X1	0.000	0.000	0.000	0.000
X2	0.000	0.000	0.000	0.000
X3	1.000	1.000	1.000	1.000
X4	1.000	1.000	.730	.418
X5	.727	.727	.865	.871
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	.470	.470	1.000	1.000
X10	1.000	1.000	.015	1.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	0.	0.	0.	0.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	56450.	56450.	66030.	63720.
W4	42610.	42610.	50410.	48530.
W5	8745.	8745.	13900.	12650.
D0	5000.	5000.	5000.	5000.
D1	7106.	7106.	27650.	12840.
D2	46150.	46150.	5000.	34500.
D3	52460.	52460.	72870.	58160.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	50687.	50687.	50682.	50687.
AVERAGE DIVIDEND	20119.	20119.	20086.	20133.
STD. DEVIATION	22710.	22710.	27400.	21899.

TABLE E-3

LINEAR MODEL

SET 1

VALUE OF C2 .5
VALUE OF C3 4.0

	VALUE OF C1			
VARIABLE	0.5	2	8	10
X1	0.000	0.000	0.000	0.000
X2	0.000	0.000	0.000	0.000
X3	1.000	1.000	1.000	1.000
X4	.722	.722	.825	.334
X5	.930	.930	1.000	1.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	.646	.646	.833	.999
X10	1.000	1.000	.027	1.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	14220.	14220.	11600.	11610.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	45400.	45400.	50260.	50250.
W4	26380.	26380.	30340.	30720.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5000.
D1	5000.	5000.	21660.	8828.
D2	38350.	38350.	5000.	30690.
D3	38350.	38350.	54910.	42160.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	81271.	81271.	81254.	81264.
AVERAGE DIVIDEND	16116.	16116.	16095.	16113.
STD. DEVIATION	17221.	17221.	20149.	16214.

TABLE E-4

LINEAR MODEL

SET 1

VALUE OF C2 .5
VALUE OF C3 5.0

	VALUE OF C1			
VARIABLE	0.5	2	8	10
X1	0.000	0.000	.063	0.000
X2	0.000	0.000	0.000	0.000
X3	1.000	1.000	1.000	1.000
X4	.581	.581	.917	.636
X5	1.000	1.000	1.000	1.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	.753	.753	.729	.673
X10	1.000	1.000	0.000	.922
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	22700.	22770.	22750.	22770.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	40210.	40200.	42410.	41050.
W4	20190.	20190.	20740.	20190.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5000.
D1	5000.	5000.	19470.	5000.
D2	34900.	34900.	5000.	33970.
D3	32120.	32110.	48350.	33970.
D4	5927.	5933.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	92354.	92425.	92405.	92425.
AVERAGE DIVIDEND	14658.	14657.	14637.	14657.
STD. DEVIATION	14634.	14631.	17501.	14960.

APPENDIX F

Selected Results of the Linear Model (Set 2)

TABLE F-1

LINEAR MODEL

SET 2

VALUE OF C2 .5
VALUE OF C3 1.0

VALUE OF C1

VARIABLE	0.5	2	8	10
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	0.000	0.000	0.000	0.000
X4	0.000	0.000	0.000	0.000
X5	0.000	0.000	0.000	0.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	10060.	10060.	10060.	10060.
V5	30290.	30290.	30290.	30290.
W0	5000.	5000.	5000.	5000.
W1	0.	0.	0.	0.
W2	35110.	35110.	35110.	35110.
W3	63950.	63950.	63950.	63950.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5000.
D1	6900.	6900.	6900.	6900.
D2	57610.	57610.	57610.	57610.
D3	63300.	63300.	63300.	63300.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	30290.	30290.	30290.	30290.
AVERAGE DIVIDEND	23801.	23801.	23801.	23801.
STD. DEVIATION	28458.	28458.	28458.	28458.

TABLE F-2

LINEAR MODEL

SET 2

VALUE OF C2 .5
VALUE OF C3 2.0

	VALUE OF C1			
VARIABLE	0.5	2	8	10
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	0.000	0.000	0.000	0.000
X4	0.000	0.000	0.000	0.000
X5	0.000	0.000	0.000	0.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	31220.	31220.	31220.	31220.
V5	53120.	53120.	53120.	53120.
W0	4991.	4991.	4991.	4991.
W1	0.	0.	0.	0.
W2	26200.	26200.	26200.	26200.
W3	45080.	45080.	45080.	45080.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5000.
D1	6909.	6909.	6909.	6909.
D2	48700.	48700.	48700.	48700.
D3	54420.	54420.	54420.	54420.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	53120.	53120.	53120.	53120.
AVERAGE DIVIDEND	28838.	28838.	28838.	28838.
STD. DEVIATION	23877.	23877.	23877.	23877.

TABLE F-3

LINEAR MODEL

SET 2

VALUE OF C2 .5
VALUE OF C3 4.0

VARIABLE	VALUE OF C1 0.5	2	8	10
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	0.000	0.000	0.000	0.000
X4	0.000	0.000	0.000	0.000
X5	0.000	0.000	0.000	0.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	60860.	60940.	60860.	60940.
V5	85130.	85190.	85130.	85190.
W0	5000.	5000.	4996.	5000.
W1	0.	0.	0.	0.
W2	13740.	13740.	13730.	13740.
W3	18630.	18600.	18620.	18600.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5000.
D1	6900.	6900.	6904.	6900.
D2	36240.	36240.	36230.	36240.
D3	41930.	41930.	41930.	41930.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	85130.	85190.	85130.	85190.
AVERAGE DIVIDEND	16678.	16678.	16677.	16678.
STD. DEVIATION	17464.	17464.	17461.	17464.

TABLE F-4

LINEAR MODEL

SET 2

VALUE OF C2 .5
VALUE OF C3 5.0

	VALUE OF C1			
VARIABLE	0.5	2	8	10
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	0.000	0.000	0.000	0.000
X4	0.000	0.000	0.000	0.000
X5	0.000	0.000	0.000	0.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	71650.	71650.	71630.	71650.
V5	96780.	96780.	96770.	96780.
W0	5000.	5000.	4998.	5000.
W1	0.	0.	0.	0.
W2	9199.	9199.	9150.	9199.
W3	9002.	9002.	9004.	9002.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
C0	5000.	5000.	5000.	5000.
C1	6900.	6900.	6912.	6900.
C2	31700.	31700.	31690.	31700.
C3	37390.	37390.	37390.	37390.
C4	5000.	5000.	5000.	5000.
C5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	96780.	96780.	96770.	96780.
AVERAGE DIVIDEND	15165.	15165.	15163.	15165.
STD. DEVIATION	15137.	15137.	15134.	15137.

APPENDIX G

Graphs of the terminal wealth, standard deviation and average dividends against c_3 for the linear model (Set 1).

NOTE: All the graphs of this Appendix have the effects of c_1 and c_2 removed.

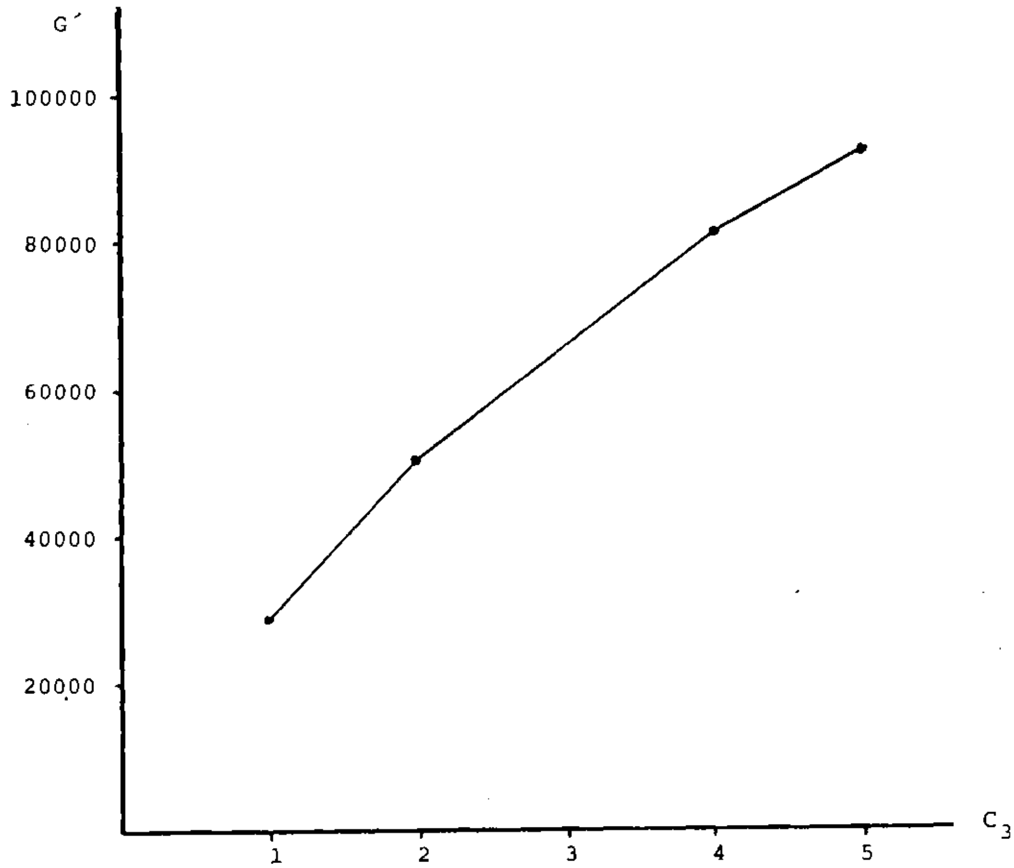

Figure G-1. Terminal Wealth vs. c_3 for Project Set 1, Linear Model.

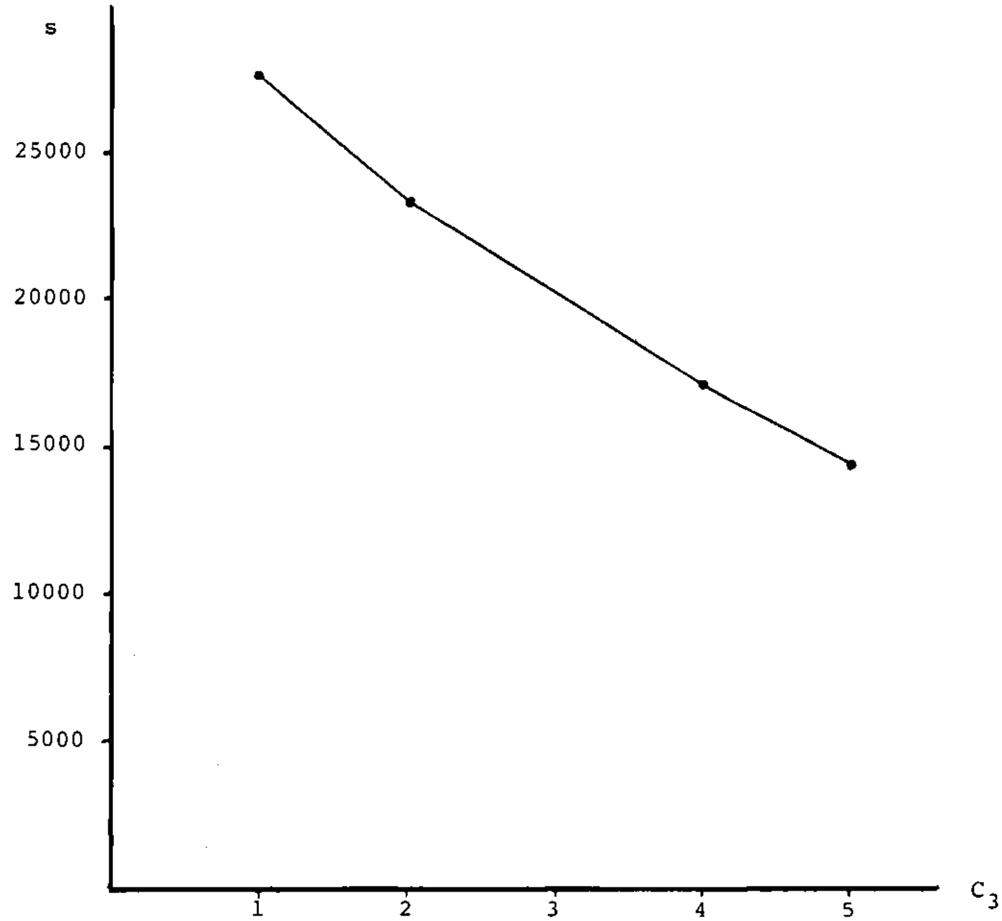

Figure G-2. Standard Deviation vs. c_3 for Project Set 1, Linear Model.

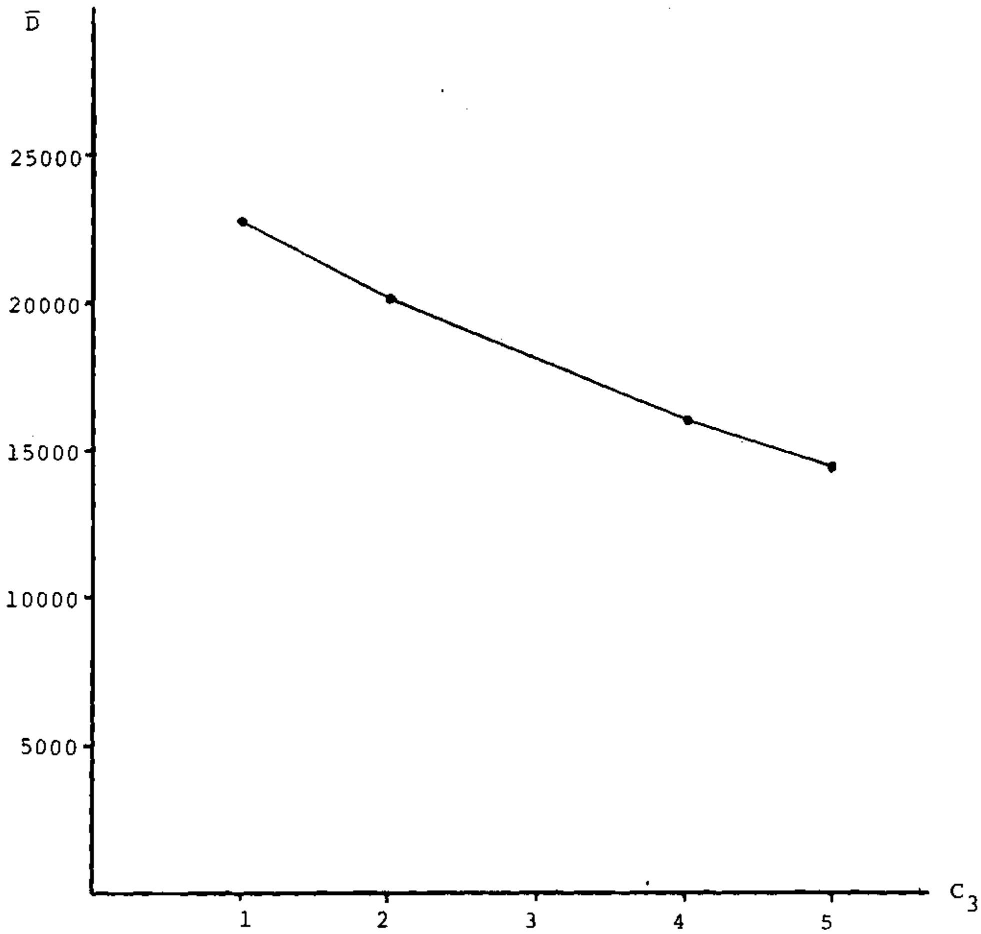

Figure G-3. Average Dividend vs. c_3 for Project Set 1, Linear Model.

APPENDIX H

Selected Results of the Nonlinear Model (Set 1)

TABLE H-1

NONLINEAR MODEL

SET 1

VALUE OF C2 .003
VALUE OF C3 10.

	VALUE OF C1			
VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	.485	.456	.452	.377
X2	0.000	0.000	0.000	0.000
X3	.902	.900	.907	.913
X4	1.000	1.000	1.000	1.000
X5	.776	.776	.780	.785
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	.202	.202	.204	.206
X9	.479	.436	.431	.412
X10	.195	.165	.164	.108
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	41714.	41532.	41363.	41514.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	9095.	8233.	10089.	10086.
W4	12180.	12012.	12429.	11943.
W5	0.	0.	0.	0.
D0	8434.	9094.	9207.	10982.
D1	10479.	11044.	10707.	11204.
D2	13364.	12629.	12427.	11153.
D3	9458.	8705.	10337.	9771.
D4	11281.	11591.	9747.	7973.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	128097.	127706.	128243.	129050.
AVERAGE DIVIDEND	9664.	9677.	9582.	9347.
STD. DEVIATION	2840.	2735.	2509.	2464.

TABLE H-2

NONLINEAR MODEL

SET 1

VALUE OF C2 .010
VALUE OF C3 10.

	VALUE OF C1			
VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	0.000	0.000	0.000	0.000
X2	0.000	0.000	0.000	0.000
X3	.485	.485	.815	4.841
X4	0.000	0.000	0.000	0.000
X5	.418	.419	.415	.417
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	0.000	.112	0.000	.112
X9	.112	.192	.112	.032
X10	.191	.033	.191	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	16875.	16901.	16866.	16846.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	2158.	2269.	2185.	2259.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5000.
D1	5000.	5000.	5000.	5000.
D2	5000.	5000.	5000.	5000.
D3	5000.	5167.	5000.	5000.
D4	5131.	5000.	5104.	5036.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	63804.	63863.	66770.	63670.
AVERAGE DIVIDEND	5022.	5028.	5017.	5006.
STD. DEVIATION	53.	68.	42.	15.

TABLE H-3

NONLINEAR MODEL

SET 1

VALUE OF C2 .003
VALUE OF C3 13.

VALUE OF C1

VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	.394	.377	.332	.210
X2	0.000	0.000	0.000	0.000
X3	1.000	1.000	1.000	1.000
X4	1.000	1.000	1.000	1.000
X5	1.000	1.000	1.000	1.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	.305	.304	.310	.322
X9	.451	.451	.449	.440
X10	.156	.135	.154	.118
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	27072.	26886.	26514.	25589.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	20513.	19875.	21442.	21607.
W4	36267.	36225.	36527.	36960.
W5	0.	0.	0.	0.
D0	5969.	6354.	6670.	8512.
D1	6875.	7260.	7473.	8750.
D2	8565.	8013.	9110.	9066.
D3	11094.	10529.	10772.	8893.
D4	13013.	13538.	10689.	7689.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	139331.	138992.	139528.	140267.
AVERAGE DIVIDEND	8418.	8449.	8285.	7985.
STD. DEVIATION	3109.	3102.	2311.	1539.

TABLE H-4

NONLINEAR MODEL

SET 1

VALUE OF C2 .006
VALUE OF C3 13.

VALUE OF C1

VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	0.000	0.000	0.000	0.000
X2	0.000	0.000	0.000	0.000
X3	.838	.832	.839	.847
X4	.535	.458	.588	.470
X5	.719	.717	.723	.730
X6	.490	.498	.470	.439
X7	0.000	0.000	0.000	0.000
X8	.191	.190	.192	.194
X9	.319	.379	.303	.435
X10	.095	.157	.077	.144
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	61325.	61961.	61532.	61041.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	5732.	6621.	5266.	5243.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	8128.	7686.	8101.	8508.
D1	6441.	6577.	6953.	8230.
D2	8299.	9245.	8489.	8634.
D3	11227.	11537.	10473.	8760.
D4	8256.	7546.	8171.	7257.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	142426.	142107.	142485.	142828.
AVERAGE DIVIDEND	7891.	7931.	7864.	7731.
STD. DEVIATION	2095.	2250.	1879.	1442.

TABLE H-5

NONLINEAR MODEL

SET 1

VALUE OF C2 .003
VALUE OF C3 20.

	VALUE OF C1			
VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	1.000	1.000	1.000	1.000
X2	0.000	0.000	0.000	0.000
X3	0.000	0.000	0.000	0.000
X4	0.000	0.000	0.000	0.000
X5	0.000	0.000	0.000	0.000
X6	1.000	1.000	1.000	1.000
X7	1.000	1.000	1.000	1.000
X8	0.000	0.000	0.000	0.000
X9	.056	.052	.058	.067
X10	.184	.173	.191	.221
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	5650.	5352.	5812.	6582.
V4	104558.	104223.	104735.	105575.
V5	162199.	161844.	162392.	163307.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	33548.	33958.	33325.	32265.
W3	0.	0.	0.	0.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	6480.	6704.	6359.	5781.
D1	5000.	5000.	5000.	5000.
D2	5000.	5000.	5000.	5000.
D3	5000.	5000.	5000.	5000.
D4	5000.	5000.	5000.	5000.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	162199.	161844.	162392.	163307.
AVERAGE DIVIDEND	5246.	5284.	5226.	5130.
STD. DEVIATION	604.	696.	554.	319.

APPENDIX I

Selected Results of the Nonlinear Model (Set 2)

TABLE T-1

NONLINEAR MODEL

SET 2

VALUE OF C2 .001
VALUE OF C3 10.

VALUE OF C1

VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	0.000	0.000	0.000	0.000
X4	0.000	0.000	0.000	0.000
X5	0.000	0.000	0.000	0.000
X6	1.000	1.000	1.000	1.000
X7	0.000	0.000	0.000	0.000
X8	0.000	0.000	0.000	0.000
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	1266.	1887.	6682.	6682.
V4	103749.	95935.	93496.	93496.
V5	157402.	154091.	149205.	149205.
W0	55000.	55000.	55000.	55000.
W1	55017.	54100.	54100.	54100.
W2	46636.	44012.	43092.	43092.
W3	0.	0.	0.	0.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
C0	5000.	5000.	5000.	5000.
C1	5917.	5000.	5000.	5000.
C2	7516.	5920.	5000.	5000.
C3	8890.	11213.	7445.	7445.
C4	9330.	14819.	25441.	25441.
C5	7877.	5988.	5000.	5000.
TERMINAL WEALTH	157402.	154091.	149205.	149205.
AVERAGE DIVIDEND	7423.	7990.	9814.	9814.
STD. DEVIATION	1683.	4079.	8203.	8203.

TABLE I-2

NONLINEAR MODEL

SET 2

VALUE OF C2 .006
VALUE OF C3 10.

VARIABLE	VALUE OF C1 1*E-7	1*E-6	1*E-5	1*E-4
X1	.571	.569	.532	.397
X2	1.000	0.000	0.000	0.000
X3	.756	.757	.751	.723
X4	.993	.955	.980	.846
X5	.705	.706	.700	.674
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	.175	.175	.173	.167
X9	.249	.287	.231	.226
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	34997.	34782.	34564.	33021.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	3383.	2207.	3227.	3066.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	6237.	6302.	6540.	7553.
D1	8759.	9309.	8541.	8318.
D2	8598.	7945.	8901.	8585.
D3	11312.	10078.	10584.	8648.
D4	9140.	10413.	8806.	7353.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	107681.	107567.	106743.	102537.
AVERAGE DIVIDEND	8172.	8174.	8062.	7576.
STD. DEVIATION	2241.	2170.	1976.	1371.

TABLE I-3

NONLINEAR MODEL

SET 2

VALUE OF C2 .010
VALUE OF C3 13.

	VALUE OF C1			
VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	.274	.274	.270	.243
X2	0.000	0.000	0.000	0.000
X3	.712	.712	.711	.704
X4	.638	.638	.627	.577
X5	.664	.664	.663	.656
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	.164	.164	.164	.162
X9	.185	.185	.192	.211
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	30027.	30027.	29904.	29561.
W0	0.	0.	0.	0.
W1	0.	0.	0.	0.
W2	0.	0.	0.	0.
W3	2744.	2744.	2760.	2747.
W4	0.	0.	0.	0.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5139.
D1	5020.	5020.	5132.	5404.
D2	6528.	6528.	6401.	6003.
D3	6379.	6379.	6349.	6020.
D4	6025.	6025.	5961.	5672.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	98506.	98506.	98261.	97184.
AVERAGE DIVIDEND	5658.	5658.	5640.	5539.
STD. DEVIATION	732.	732.	672.	432.

TABLE I-4

NONLINEAR MODEL

SET 2

VALUE OF C2 .005
VALUE OF C3 16.

VALUE OF C1

VARIABLE	1*E-7	1*E-6	1*E-5	1*E-4
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	1.000	1.000	1.000	1.000
X4	0.000	0.000	0.000	0.000
X5	1.000	1.000	1.000	1.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	.312	.310	.314	.318
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	41367.	41622.	41117.	43830.
W0	20623.	20598.	20901.	21888.
W1	53293.	53617.	53387.	54481.
W2	67856.	67775.	68495.	70310.
W3	74149.	73303.	75091.	76773.
W4	28706.	28368.	29045.	29535.
W5	0.	0.	0.	0.
D0	5623.	5598.	5901.	6888.
D1	7685.	8047.	7477.	7466.
D2	6344.	5888.	6867.	7456.
D3	8148.	7513.	8246.	7636.
D4	9998.	10848.	9026.	7124.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	151041.	151018.	151082.	151392.
AVERAGE DIVIDEND	7133.	7149.	7086.	6829.
STD. DEVIATION	1843.	2155.	1487.	982.

TABLE I-5

NONLINEAR MODEL

SET 2

VALUE OF C2 .010
VALUE OF C3 20.

VARIABLE	VALUE OF C1 1*E-7	1*E-6	1*E-5	1*E-4
X1	1.000	1.000	1.000	1.000
X2	1.000	1.000	1.000	1.000
X3	1.000	1.000	1.000	1.000
X4	0.000	0.000	0.000	0.000
X5	1.000	1.000	1.000	1.000
X6	0.000	0.000	0.000	0.000
X7	0.000	0.000	0.000	0.000
X8	.337	.337	.339	.343
X9	0.000	0.000	0.000	0.000
X10	0.000	0.000	0.000	0.000
V0	0.	0.	0.	0.
V1	0.	0.	0.	0.
V2	0.	0.	0.	0.
V3	0.	0.	0.	0.
V4	0.	0.	0.	0.
V5	47123.	47123.	46972.	46569.
W0	20000.	20000.	20000.	20253.
W1	50702.	50702.	50689.	51034.
W2	63622.	63622.	63893.	64967.
W3	69318.	69318.	69839.	71184.
W4	24981.	24981.	25199.	25805.
W5	0.	0.	0.	0.
D0	5000.	5000.	5000.	5253.
D1	5802.	5802.	5789.	5851.
D2	5000.	5000.	5285.	5973.
D3	6510.	6510.	6677.	6509.
D4	8585.	8585.	8037.	6596.
D5	5000.	5000.	5000.	5000.
TERMINAL WEALTH	160380.	160380.	160436.	160659.
AVERAGE DIVIDEND	5982.	5982.	5958.	5864.
STD. DEVIATION	1412.	1412.	1191.	646.

APPENDIX J

Graphs of the terminal wealth, standard deviation and average dividend against each one of the parameters for the non-linear model.

NOTE: The graphs of this Appendix (except J-1, J-4, and J-7) have the effects of c_1 removed.

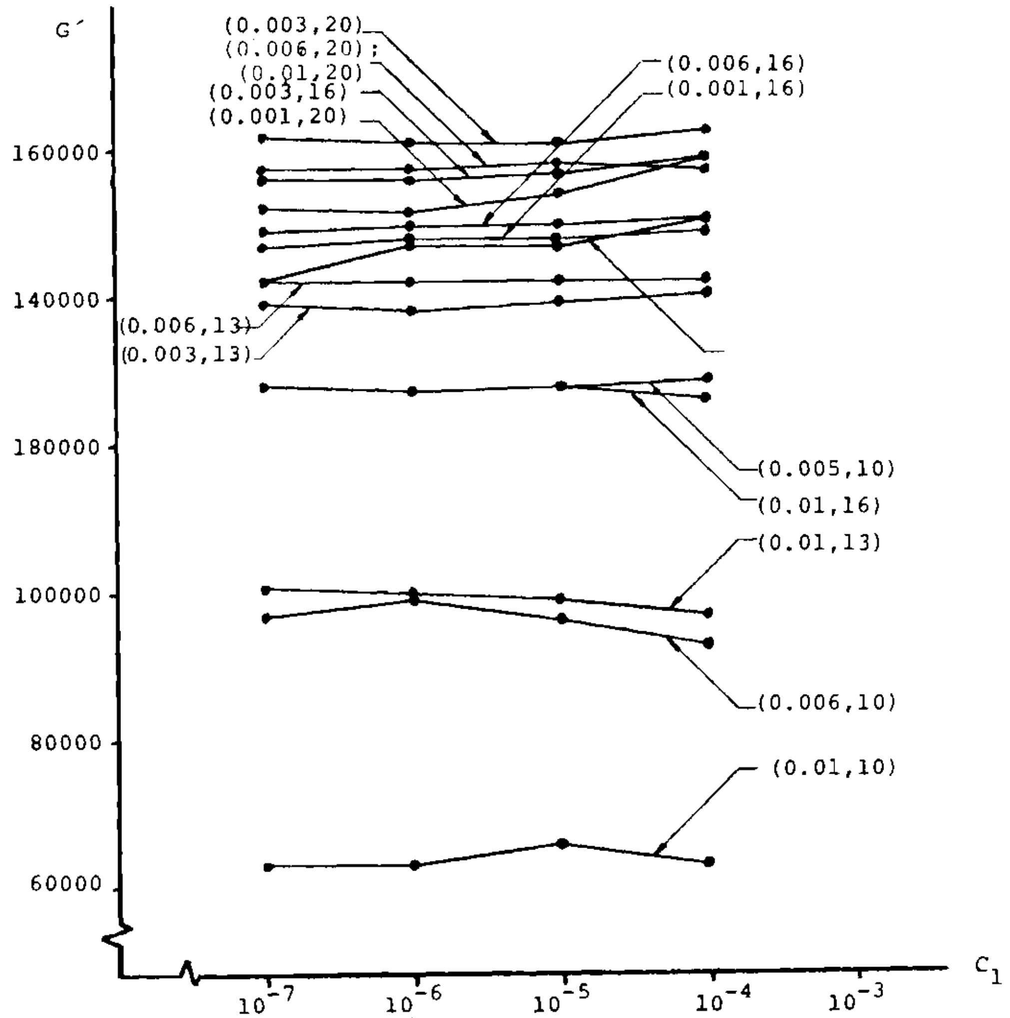

Figure J-1. Terminal Wealth vs. c_1 for Project Set 1, Nonlinear Model.

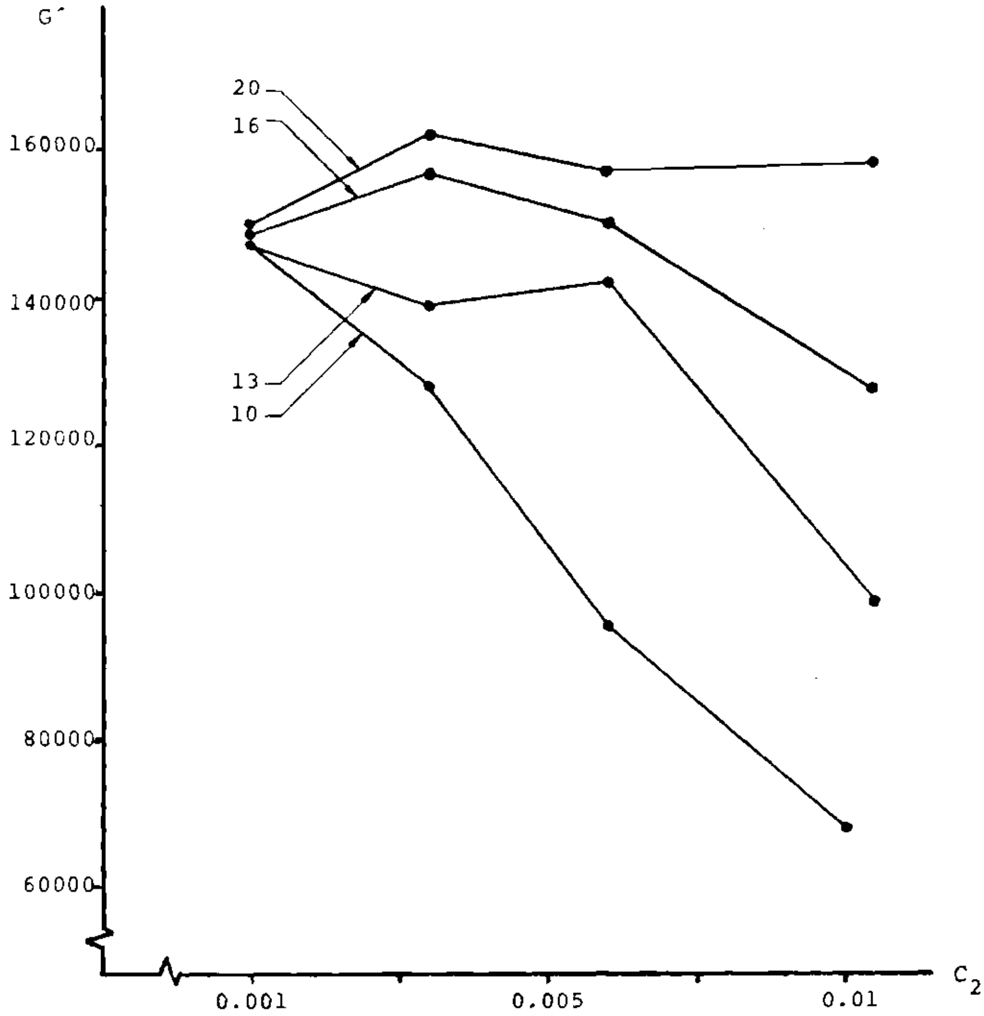

Figure J-1. Terminal Wealth vs. c_2 for Project Set 1, Nonlinear Model.

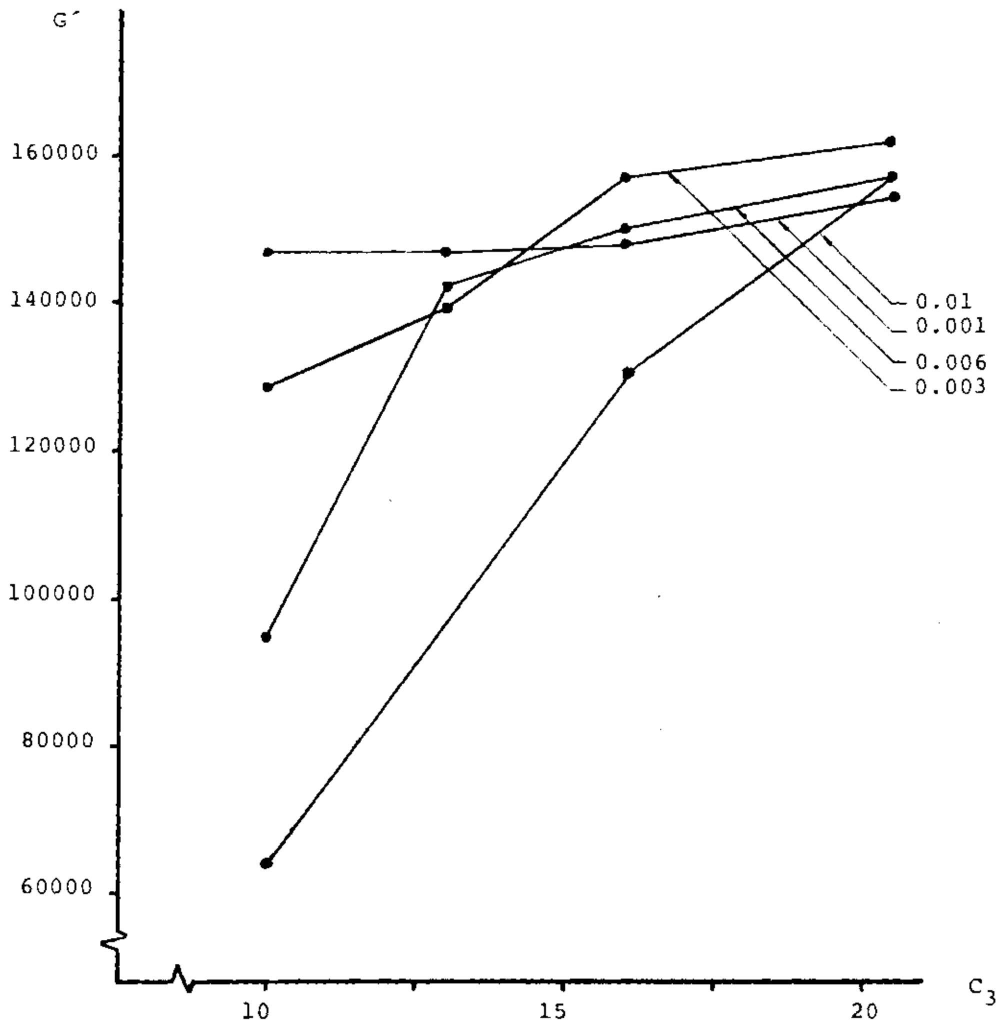

Figure J-3. Terminal Wealth vs. c_3 for Project Set 1, Nonlinear Model.

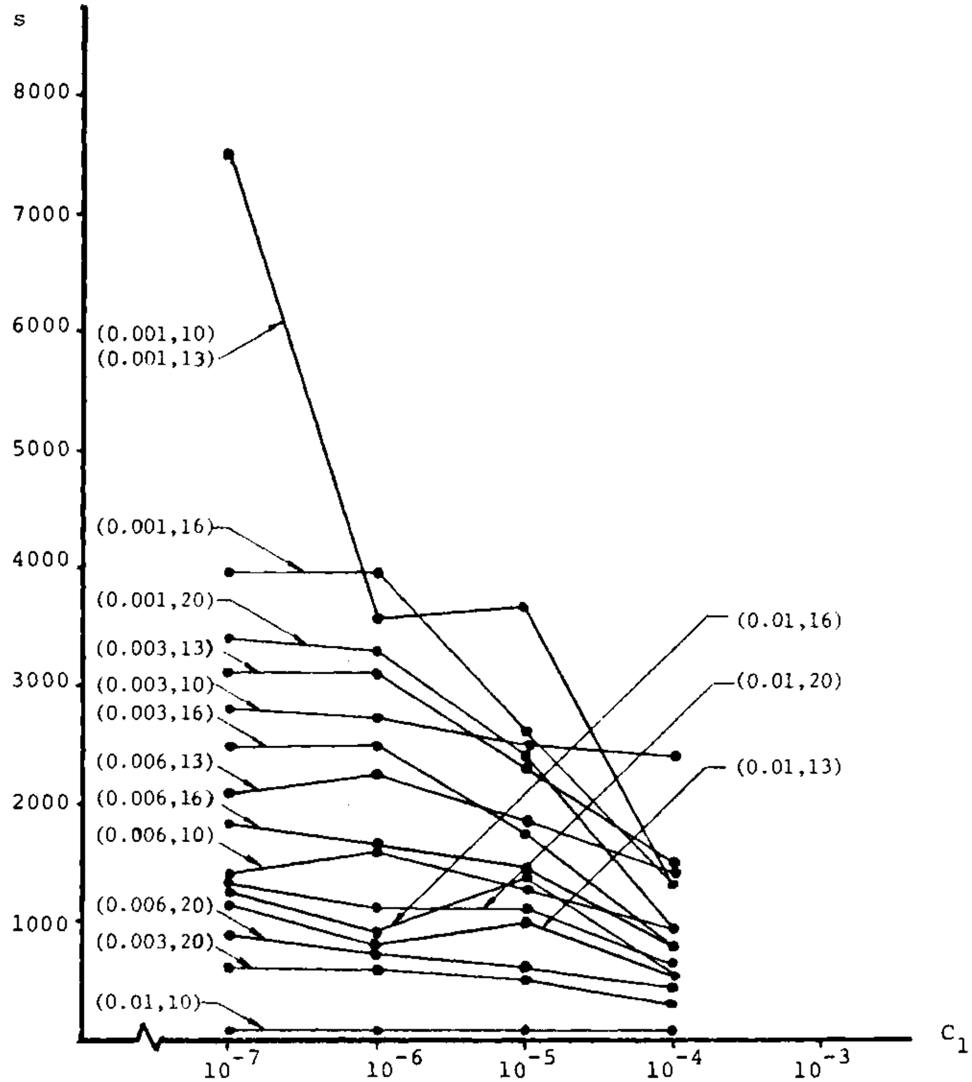

Figure J-4. Standard Deviation vs. c_1 for Project Set 1, Nonlinear Model.

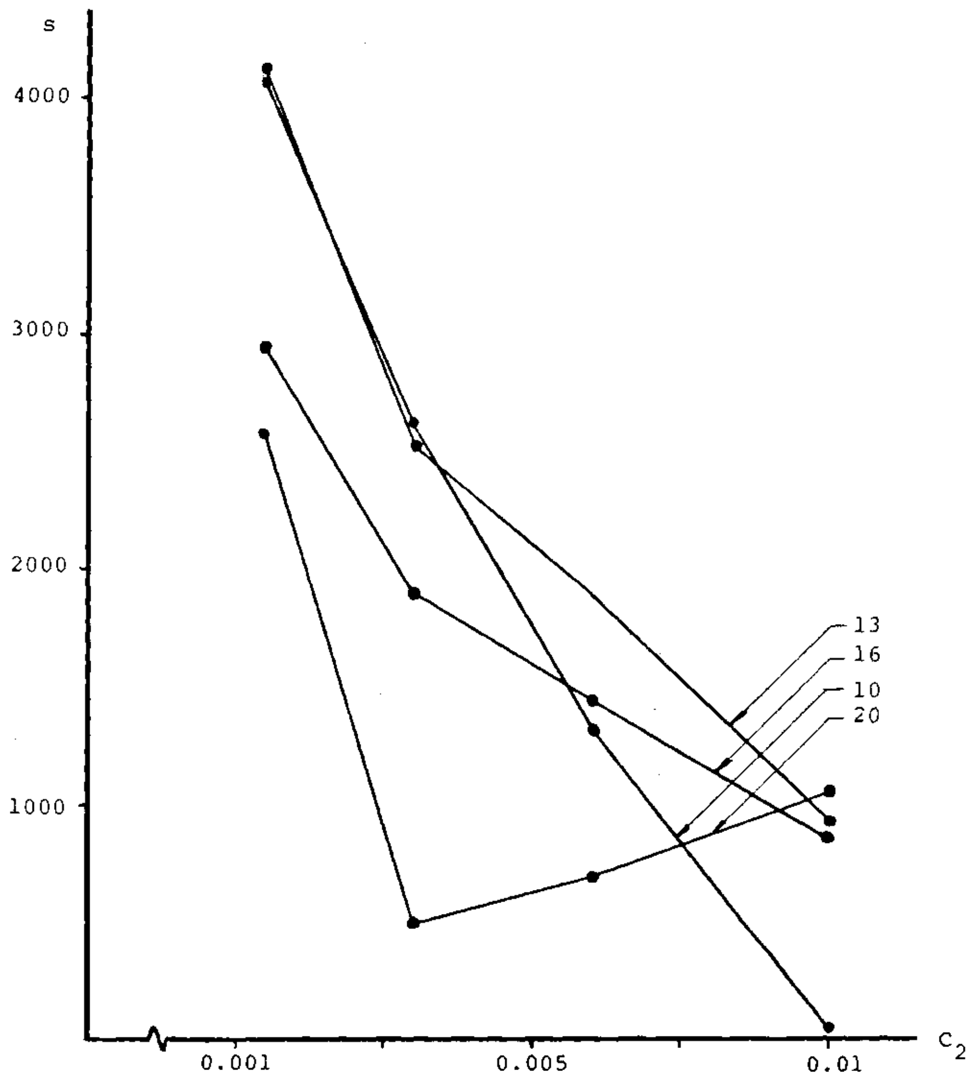

Figure J-5. Standard Deviation vs. c_2 for Project Set 1, Nonlinear Model.

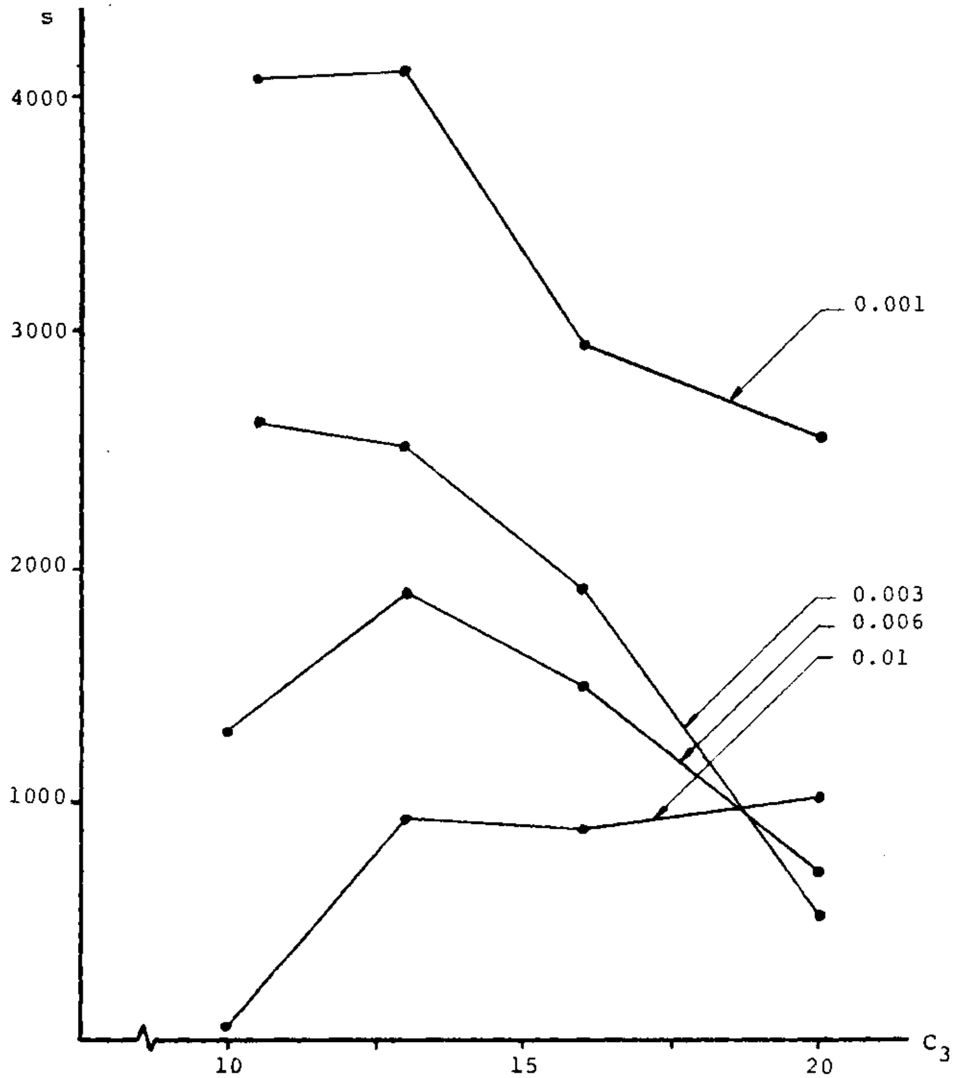

Figure J-6. Standard Deviation vs. c_3 for Project Set 1, Nonlinear Model.

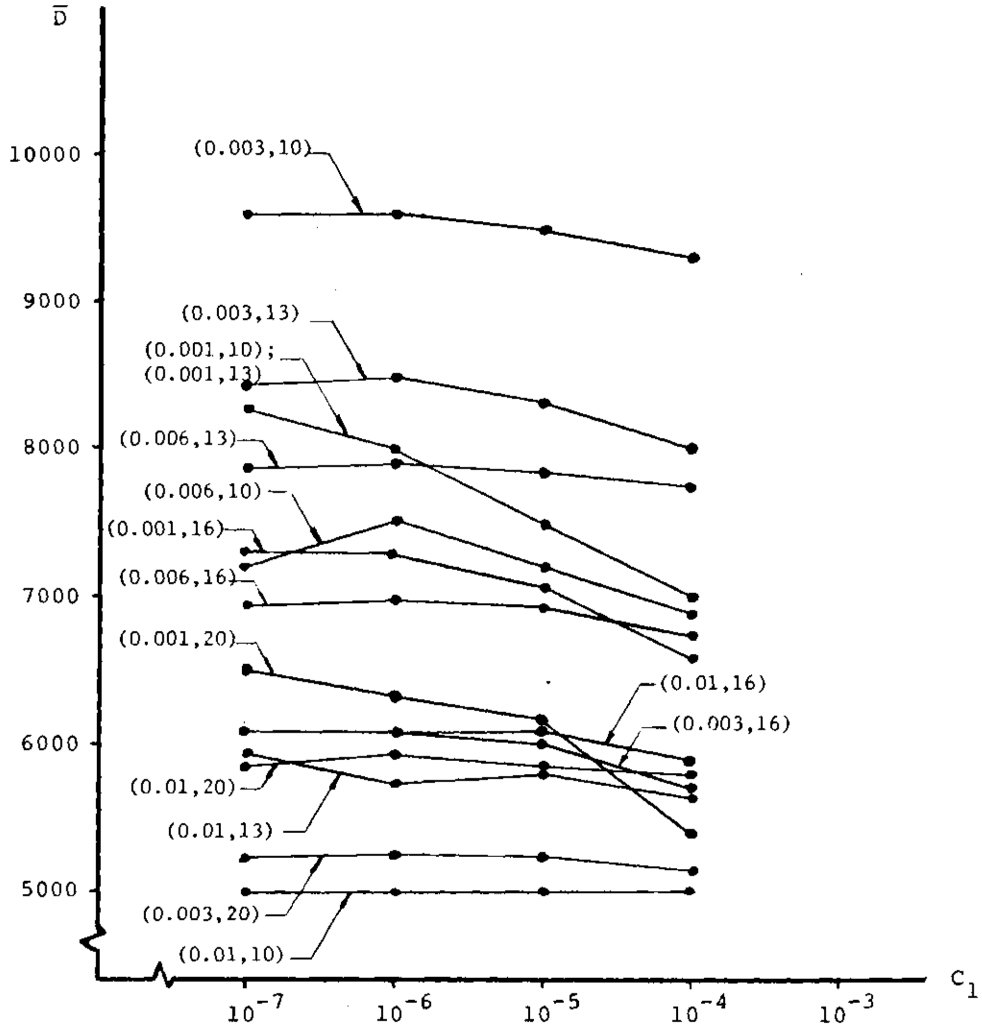

Figure J-7. Average Dividend vs. c_1 for Project Set 1, Nonlinear Model.

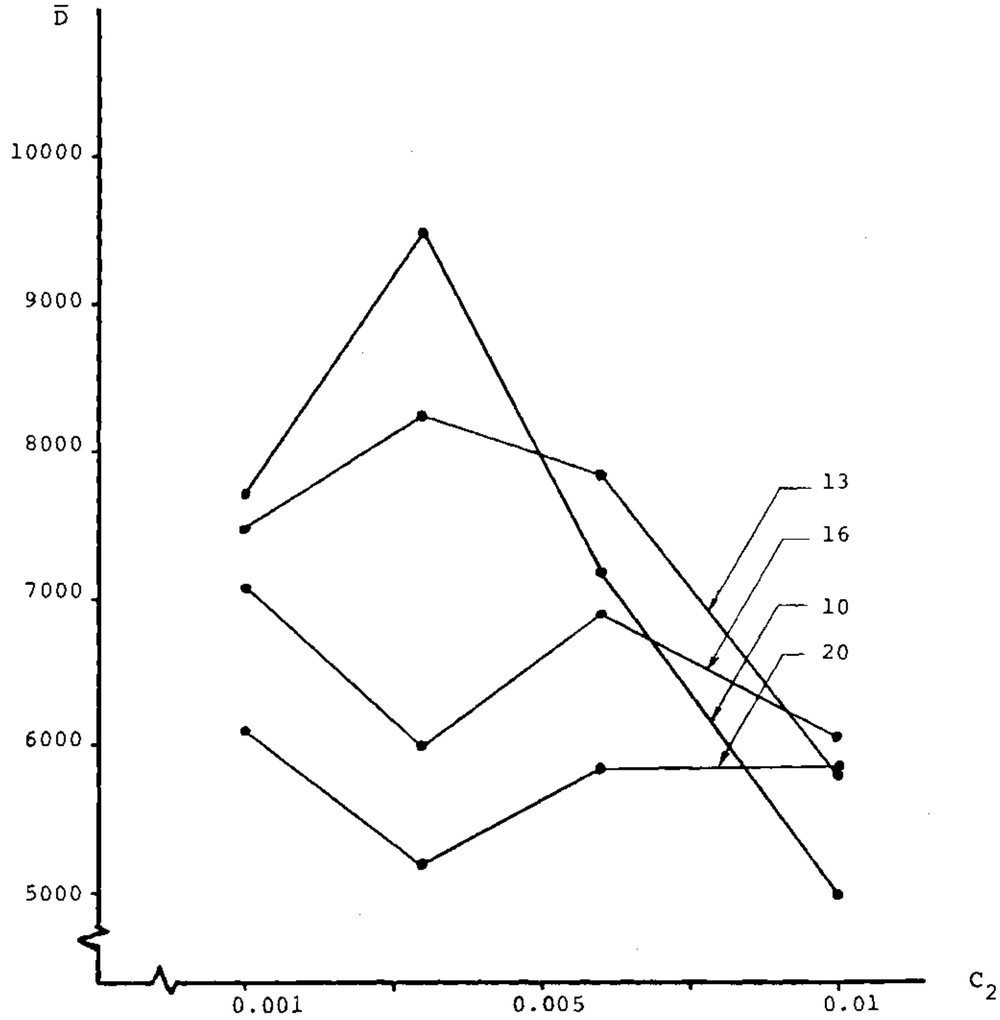

Figure J-8. Average Dividend vs. c_2 for Project Set 1, Nonlinear Model.

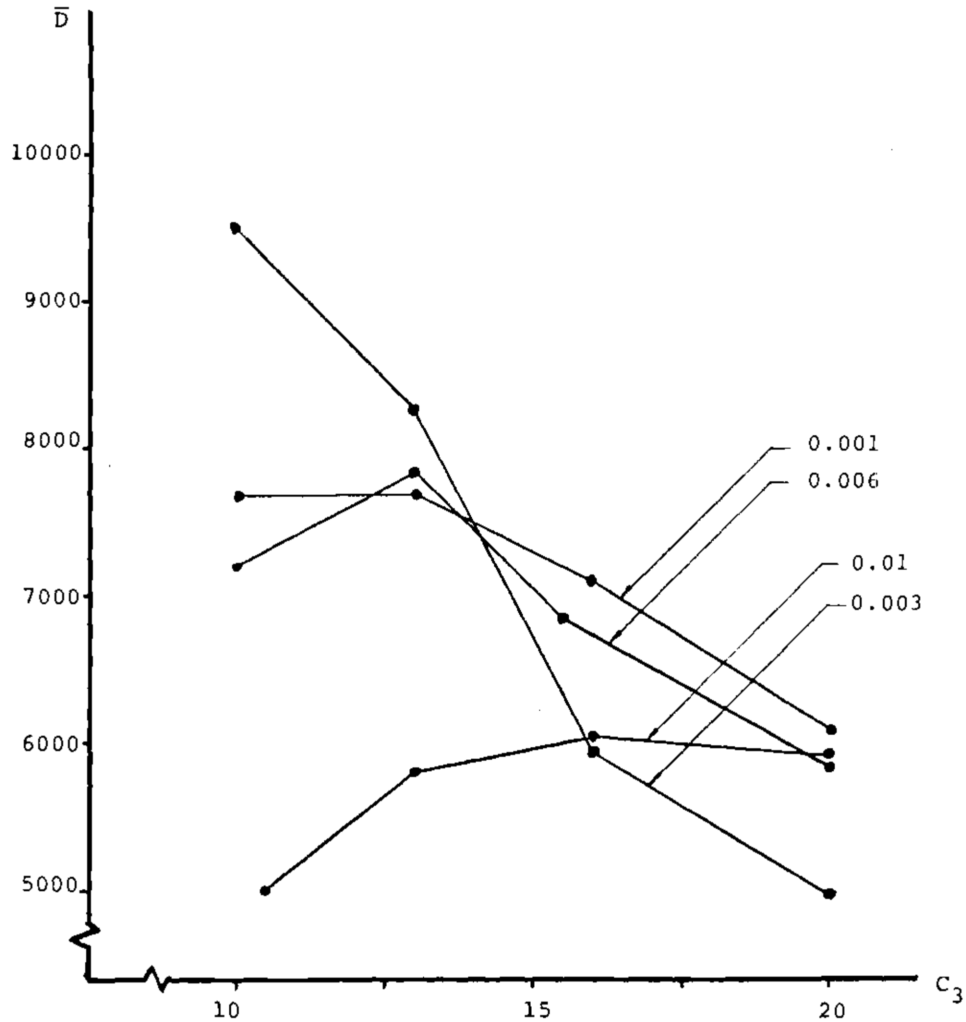

Figure J-9. Average Dividend vs. c_3 for Project Set 1, Nonlinear Model.

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